M332 BN

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

M332 Complex Analysis
Broadcast Notes

SUP 12107 0

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Broadcast Notes

M332 Television Programme 0

Real Analysis Review.

INTRODUCTION (to be read BEFORE WATCHING THE PROGRAMME.)

Before watching the programme you should have read the following sections of Unit 0:

pp 8-11; pp 33-35; p41, down to end of the Remark.

The programme contains certain algebraic manipulations we would like you to be familiar with before you watch, so complete the following examples:-

1. (a) Show that $|x^2 - 4| \leq |x - 2|\left(|x - 2| + 4\right)$.
   (b) If further $|x - 2| < 1$, what can you say about $|x^2 - 4|$?
2. If $|x - 2| < 1$ what can you say about
   (a) $|x + 1|$ (lower bound)?
   (b) $\dfrac{1}{|x + 1|}$ (upper bound)?
3. Use the Triangle Inequality (cunning form) to show that $|x + 1| \geq ||x - 2| - |-3||$. Then if $|x - 2| < 1$, show that $|x + 1| > 2$.

Programme Review

The programme is designed as a fairly straightforward tutorial which will introduce you to the so-called $\varepsilon - \delta$ approach to limits (and, later, continuity). This was the approach adopted by M231 (the original course in Analysis preceding M332). However M231 has been withdrawn and you may have studied M203 instead. M203 approached limits via 'nests'. The programme is part of the bridging material to lead you to the $\varepsilon - \delta$ approach.

First the programme reviews some limits; $\lim_{x\to 2} x^2 = 4$ may seem obvious,

but what is $\lim_{x\to 0} \frac{\sin x}{x}$? or $\lim_{x\to 0} \left(x^2 \sin \frac{1}{x}\right)$?

Graphs lead us to <u>suspect</u> that they have limits 1 and 0 respectively, but the important thing is to <u>prove</u> that these are the limits.

The remainder of the programme concerns the meaning and techniques behind the definition:

> The function f has limit ℓ near a if
> to every given $\varepsilon > 0$ there corresponds some $\delta > 0$ such that
> if $0 < |x - a| < \delta$ then $|f(x) - \ell| < \varepsilon$.

Don't forget that f may not be defined at x = a; hence the condition $0 < |x - a|$. The techniques demonstrated in the remainder of the programme revolve around the idea of a "challenge";

1. An "ε" is given to show how close to "ℓ" f(x) has to be.
2. The challenge is to find a "δ" such that $0 < |x - a| < \delta$ <u>guarantees</u> that $|f(x) - \ell| < \varepsilon$.

Naturally ε will have to be as small as you like and there will inevitably be a connection between the "ε" given and the "δ" found - in fact δ will have to be expressed in rows of ε. The first example is to show that $\lim_{x\to 2} x^2$ equals 4, in terms of the $\varepsilon - \delta$ definition of limit. The proof developed in the programme is as follows:

Let $\varepsilon > 0$ be <u>given</u>.

Then

$$|x^2 - 4| = |x - 2||x + 2|$$
$$= |x - 2||(x - 2) + 4|$$
$$\leq |x - 2|(|x - 2| + 4)$$

$< \frac{\varepsilon}{5}$ if $|x - 2| < \frac{\varepsilon}{5}$; < 5 if $|x - 2| < 1$

$$< \tfrac{\varepsilon}{5}.5 \quad \text{if } |x - 2| < \text{both } \tfrac{\varepsilon}{5} \text{ and } 1$$
$$= \varepsilon \quad \text{if } |x - 2| < \delta, \text{ where } \delta = \text{Min}\left(\tfrac{\varepsilon}{5}, 1\right).$$

Notice the technique of expressing $|x^2 - 4|$ in terms of $|x - 2|$ to connect with δ. Also notice how this uses the first part of the

> Triangle Inequality: $|x + y| \leq |x| + |y|$
> $|x - y| \geq ||x| - |y||$

The second part of the triangle inequality is used in the next problem:-

<u>Problem</u>. Prove that if $f(x) = \frac{1}{x + 1}$, then $\lim_{x\to 2} f(x) = \frac{1}{3}$.

<u>Proof</u> Let $\varepsilon > 0$ be <u>given</u>. Then:

$$|f(x) - \tfrac{1}{3}| = |\tfrac{1}{x + 1} - \tfrac{1}{3}|$$
$$= \frac{1}{3|x + 1|} \cdot |2 - x|.$$

about $\frac{1}{9}$ near 2 ; small

If $|x - 2| < 1$, then $|x + 1| = |(x - 2) + 3|$
$$\geq ||x - 2| - |-3||$$
$$= 3 - |x - 2|$$
$$\geq 3 - 1 = 2.$$

So if <u>both</u> $|x - 2| < 1$ <u>and</u> $|x - 2| < 6\varepsilon$, then

$$|f(x) - \tfrac{1}{3}| < \frac{1}{3.2}\,.6\varepsilon = \varepsilon.$$

i.e. the definition of $\lim_{x\to 2} f(x) = \frac{1}{3}$ holds with the choice $\delta = \text{Min}(1, 6\varepsilon)$.■

Most proofs of limits that you'll come across are rather like this.

To show that $\lim_{x\to a} f(x) = \ell$, first work out $|f(x) - \ell|$; it often has a factor $|x - a|$.

So if you write $|f(x) - \ell|$ as $|x - a| \times$ (some function $g(x)$), first find a constant K such that $|g(x)| < K$ if $|x - a|$ is less than 1 (or any convenient fixed number). Then if you choose $\delta = \min(1, \frac{\varepsilon}{K})$, you have that $|f(x) - \ell| < \frac{\varepsilon}{K}$. $K = \varepsilon$, as required.

Finally the programme looks at the connection between limits and continuity:

If $f(x)$ has limit ℓ near a and in addition

(i) f is defined at a, and

(ii) $f(a) = \ell$,

then f is continuous at a. This coincides exactly with the intuitive idea that the graph of f 'has no break' at a; and can be formulated quite analogously to the definition of limit:

> The function f is continuous at a if:
>
> to every given $\varepsilon > 0$, there corresponds some $\delta > 0$ such that
>
> if $|x - a| < \delta$ then $|f(x) - f(x)| < \varepsilon$.

Post-Programme Exercise

(1) Prove that $\lim_{x\to -1} \frac{x + 2}{x^2 + 1} = \frac{1}{2}$.

(ii) Deduce that the function $\frac{x + 1}{x^2 + 1}$ is continuous at -1.

Solution

(i) Let $\varepsilon > 0$ be given,

$$\left|\frac{x + 2}{x^2 + 1} - \frac{1}{2}\right| = \frac{|2x + 4 - x^2 - 1|}{2|x^2 + 1|}$$

$$= \frac{|3 + 2x - x^2|}{2|x^2 + 1|}$$

$$= \frac{|3 - x|}{2|x^2 + 1|} \cdot |1 + x|$$

$= |x - (-1)|$

If $|x - (-1)| < 1$, then:

$$|3 - x| = |4 - (x + 1)| \leq 4 + |x + 1| < 4 + 1 = 5;$$

and $|x^2 + 1| \geq 1$.

Hence if $\delta = \text{Min}\,(1, \frac{2\varepsilon}{5})$, and $|x - (-1)| < \delta$,

then $\left|\frac{x + 2}{x^2 + 1} - \frac{1}{2}\right| < \frac{5}{1.2} \cdot \frac{2}{5}\varepsilon = \varepsilon$, as required.

(ii) The function $\frac{x + 2}{x^2 + 1}$ is defined at -1. Its value there is $\frac{1}{2}$, i.e. equal to $\lim_{x\to -1} \frac{x + 2}{x^2 + 1}$. Hence the function is continuous at -1.

Radio Broadcast Notes

M332 Radio Programme 1

Continuous Functions and Compact Sets

INTRODUCTION (to be read BEFORE LISTENING TO THE PROGRAMME)

This programme concentrates on some of the more difficult ideas of Unit 2. Before listening to it you should be familiar with some basic definitions. To remind yourself of these, if you have time, you might like to spend a minute or two writing down the definitions of the following terms (where C and R are being used to represent the complex numbers and respectively).

1. The function f is continuous at the point a ($a \in \mathbb{C}$).
2. S is an open set in $\mathbb{C}$.
3. S is a closed set in $\mathbb{C}$.
4. T is the complement of the set S in $\mathbb{C}$.
5. T is the inverse image $f^{-1}(S)$ of the set S in $\mathbb{C}$.

The solutions to 1,2,3 are given as Notes 1,2 and 4.
in the attached broadcast notes for TV1. The solution to 4 is,

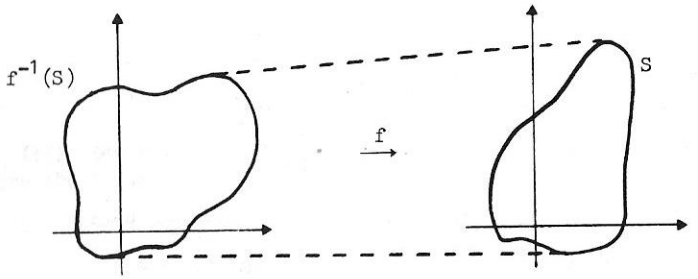
$$T = \mathbb{C} - S = \{z : z \notin S\}.$$

You should recall - that a set is closed if and only if its complement is open.
(Theorem 5 of Unit 2).

The solution to 5 is

$$f^{-1}(S) = \{z \in \text{domain of } f : f(z) \in S\}.$$

Notice that S is a set in the range of the function f and $f^{-1}(S)$ is a SET in the domain of f. It is made up of all those points z that are mapped into S by the function f (i.e. such that $f(z) \in S$).



The following notes summarize the main points of the programme and are intended for use during the programme. To make it easy to follow them, the references used in the programme - such as item numbers, and figure numbers - are given in the margin.

We suggest that immediately after the programme you attempt the self-assessment questions - which come after the notes - to help consolidate the ideas of the programme.

NOTES (to be followed DURING THE PROGRAMME)

Item 1 Part of Theorem 6 of Unit 2 states that the inverse image of any open set under a continuous function must itself be open.

Figure 1

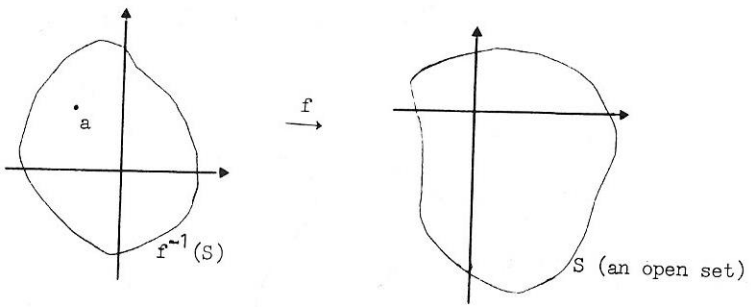


Figure 2

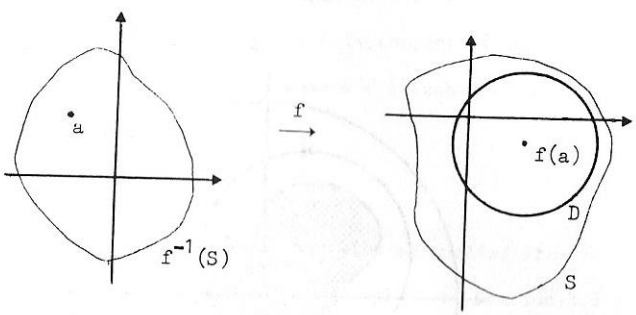
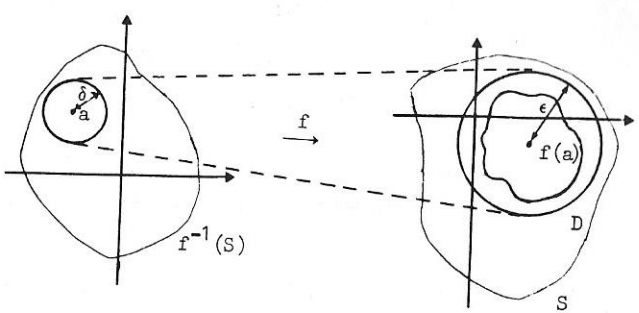


Figure 3



Item 2 Part of the Corollary to Theorem 6 of Unit 2 states that if f is a real-valued function continuous at each point of $\mathbb{C}$ and if S is an open interval of $\mathbb{R}$, then $f^{-1}(S)$ is open.

This corollary can be used to establish that certain subsets of $\mathbb{C}$ are open sets.

Example: Show that $\{z: |z-1| + |z+1| > 2\}$ is an open set.

Consider the function f defined by $f: z \to |z-1| + |z+1|$.

f is a continuous function (see Self Assessment Question 2 below) from C to R. Let $S = \{x : x \in R \text{ and } x > 2\}$

Then S is open in $\mathbb{R}$ and $f^{-1}(S) = \{z : f(z) > 2\}$

$= \{z : |z-1| + |z+1| > 2\}$.

Therefore, by the above corollary, the given set is open in $\mathbb{C}$.

(The given set is, in fact, the outside of an ellipse, but it is not necessary to discover this in order to show that the set is open.)

Item 3 Compact Sets

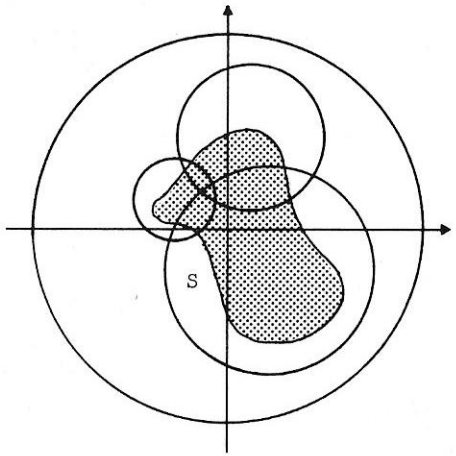
The definition of compactness states that a set S in $\mathbb{C}$ is compact if, no matter how an open disc D_a, centred at a, is assigned to each point a of S, there are a finite number of elements $a_1, a_2, \ldots, a_n$ of S such that $S \subseteq D_{a_1} \cup \ldots \cup D_{a_n}$.

Item 4 A compactness argument

Theorem 10 of Unit 2 states that any compact set is bounded and closed.

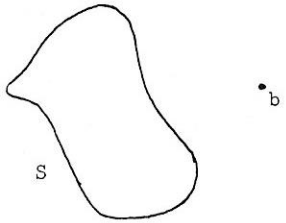
To prove this, let S be any compact set.

(i) To prove S is bounded



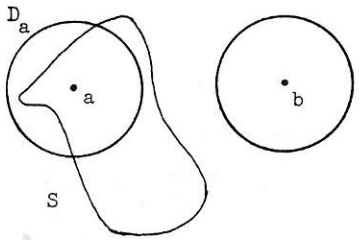
(ii) To prove S is closed

Figure 1



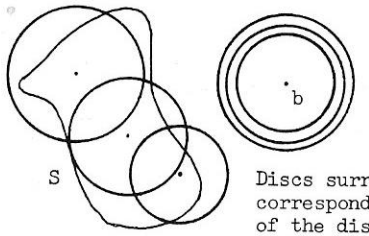
b is any point in the complement of S

Figure 2



a is any point in S. The circles are of radius $\frac{1}{3}|a - b|$.

Figure 3



Discs surrounding b, corresponding to each of the discs around points in S.

One open disc for each point in S - covering all the points of S.

S is compact, so a finite set $\{D_{a_1},\ldots,D_{a_n}\}$ of the discs covers S. Correspondingly there is a finite set of discs $\{D_{a_1,b},\ldots,D_{a_n,b}\}$ around b.

Let $D_{a,b}$ be the smallest of these. *

Then $D_{a,b} \cap D_{a_i} = \phi$ for $1 \leq i \leq n$; so $D_{a,b} \cap S = \phi$; hence b is an interior point of $\mathbb{C} - S$, and so $\mathbb{C} - S$ must be open.

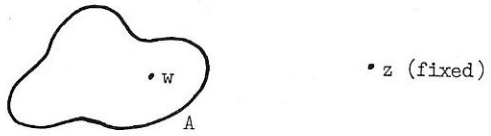
Therefore S must be closed.

Item 5 One application of the result that a continuous function on a compact set is bounded and attains its bounds is the definition of the distance from a point to a compact set.

Let A be a compact set, and let $z \notin A$.

Then the distance from z to A is defined by

$$d_A(z) = \min\{|z-w| : w \in A\}.$$



To show that $d_A(z)$ is well defined:

Let f be the function defined by $f(w) = |z-w|$, $w \in A$.

f is continuous and, since A is compact, f must attain its bounds; i.e. there is some $w \in A$ such that $|z-w| \leq |z-a|$ for all $a \in A$.

Then $|z-w| = d_A(z)$.

N.B. A similar argument can also be applied if A is closed but unbounded. (See Example 2 of Section 2.9.)

Self-Assessment Questions

1. The following questions refer to Item 1 in the Notes.

   (i) How do we know in Fig. 2 that there exists a disc D, centred at f(a), such that $D \subseteq S$?

   (ii) What property of f is illustrated by Fig. 3?

   (iii) What have we therefore demonstrated in Fig. 3?

2. Show that the function f defined in Item 2 of the Notes is continuous.

3. State the definition of compactness.

4. (i) Outline the proof that a compact set is closed.

   (ii) What are the values of r_a, the radius of the open disc D_a, and r_b, the radius of the corresponding open disc centred at b in Fig. 2 of Item 4 of the Notes?

   (iii) Where in Item 4 do we use the result that a finite set of numbers always has a least member?

5. Which of the following statements must be true?

   (i) A continuous function on a bounded set is bounded.

   (ii) A continuous function on a compact set is bounded.

   (iii) A continuous bounded function on C attains its bounds.

   (iv) A continuous function on a compact set may not attain its bounds.

6. Let A be an open set and $z \notin A$.

   If the distance, $d_A(z)$ is defined by

   $$d_A(z) = \min\{|z-w| : w \in A\},$$

   is d_A well defined? Give a proof or a counter-example.

Solutions

1. (i) It follows since S is an open set.

   (ii) The continuity of S.

   (iii) We have demonstrated that $f^{-1}(S)$ is open

2. $f(z) = |z-1| + |z+1|$.

   Let $g(z) = |z|$. We want to show that, for any a, g is continuous at a. Given any $\varepsilon > 0$, if $|z-a| < \varepsilon$, then

$$\begin{aligned} |g(z) - g(a)| &= ||z| - |a|| \\ &\leq |z-a| \\ &< \varepsilon. \end{aligned}$$

   Therefore g is continuous.

   Now $f = g \circ (z \to z - 1) + g \circ (z \to z + 1)$ and so, by Theorems 2 and 3 of Unit 2, f is a continuous function

3. See Item 3 in the Notes.

4. (i) See (ii) of Item 4 of the Notes.

   (ii) $r_a = r_b = \frac{1}{3}|b-a|$. (Any numbers such that $r_a + r_b < |b-a|$ would do).

   (iii) At the line marked *.

5. Only (ii).

6. $d_A(z)$ is not well defined since $\{|z-w| : w \in A\}$ may not have a least member.

   For instance, let $A = \{z : |z| < 1\}$ and let $z = 1+0i$.

Radio Broadcast Notes

M332 Radio Programme 2

Complex Integration

INTRODUCTION (to be read BEFORE LISTENING TO THE PROGRAMME)

This programme is about complex integration, and in particular about the integration of functions along arcs, directed paths and contours. We shall explain the differences between these terms, and will illustrate them with some specific examples.

The programme assumes the results of some integrations of complex-valued functions (of a real variable). The method for doing these is explained in Section 4.1. However, we list the ones used here, so that if you have a few minutes before the programme you might like to work through them.

(i) $$\int_0^{2\pi} i\, e^{4it}\, dt = \frac{i}{4i} e^{4it} \Big|_0^{2\pi} = \frac{1}{4}(e^{8i\pi} - e^0) = 0.$$

(ii) $$\int_0^1 (1+i)^4 t^3 dt = (1+i)^4 \frac{1}{4} t^4 \Big|_0^1 = \frac{1}{4}(1+i)^4.$$

(iii) $$\begin{aligned} \int_0^1 (t^2 + it^2)^3 (2t + 2it)dt &= \int_0^1 2t^7 (1+i)^4 dt \\ &= (1+i)^4 2 \cdot \frac{1}{8} t^8 \Big|_0^1 \\ &= \frac{1}{4}(1+i)^4. \end{aligned}$$

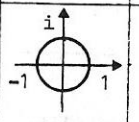
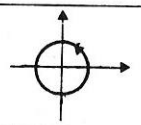
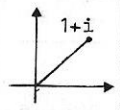
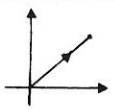
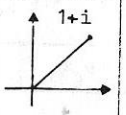
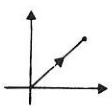
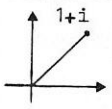
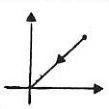
(iv) $$\begin{aligned} \int_0^1 [(1-t)+i(1-t)]^3(-1-i)dt &= \int_0^1 -(1+i)^4 (1-t)^3 dt \\ &= (1+i)^4 \cdot \frac{1}{4}(1-t)^4 \Big|_0^1 \\ &= -\frac{1}{4}(1+i)^4. \end{aligned}$$

You should refer to the following notes while listening to the programme. The self assessment questions, which come after the notes, are intended to test your understanding of the programme, and we suggest that you attempt them immediately the programme is finished.

NOTES (to be followed DURING THE PROGRAMME)

1. Complex-Valued Functions (Section 4.1). A complex-valued function (of a real variable) is a function from a subset of $\mathbb{R}$ to $\mathbb{C}$.
2. Arcs (Section 4.3). An arc is a continuous complex-valued function whose domain is a closed interval [a,b].

Table 1:

	Arc	Path	Directed Path
Example 1:	$t \to e^{it}$, $0 \le t \le 2\pi$		
Example 2:	$t \to t+it$, $0 \le t \le 1$		
Example 3:	$t \to t^2+it^2$, $0 \le t \le 1$		
Example 4:	$t \to (1-t)+i(1-t)$, $0 \le t \le 1$		

Given a path with a certain parametrization, a reparametrization of the path is simply a different arc with the same path, corresponding to a change of variable. If the change of variable corresponds to an increasing continuously differentiable function, then the direction of the path stays the same, and we have a proper reparametrization. (See Section 4.3.) (Strictly speaking the derivative should not vanish, but this condition is not always insisted on.)

3. Types of Arc (See Section 4.3.)

A closed arc is one in which the end points coincide.

A simple arc is one which is one-one.

A simple-closed arc is a closed arc which is one-one except for the end points.

A smooth arc is a differentiable arc whose derivative is continuous and non-zero. (If the arc γ, with domain [a,b], is closed, we require also that $\gamma'(a) = \gamma'(b)$.)

Table 2:

	Arc	Closed?	Simple?	Simple-closed?	Smooth?
Example 1:	$t \to e^{it}$, $0 \le t \le 2\pi$	Yes	No	Yes	Yes
Example 2:	$t \to t+it$, $0 \le t \le 1$	No	Yes	No	Yes
Example 3:	$t \to t^2+it^2$, $0 \le t \le 1$	No	Yes	No	No
Example 4:	$t \to (1-t)+i(1-t)$, $0 \le t \le 1$	No	Yes	No	Yes

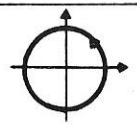
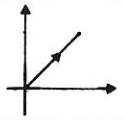
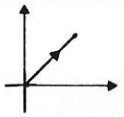
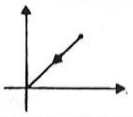
4. Integration along smooth arcs (See Section 4.5.)

If f is a continuous function and γ is a smooth arc with domain $[a,b]$, then the integral of f along γ is defined by

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Example: Let $f: z \to z^3$.

Table 3

	Arc, γ	$\int_\gamma z^3 dz$	Directed Path
Example 1.:	$t \to e^{it}$, $0 \le t \le 2\pi$	$\int_0^{2\pi} e^{3it}.ie^{it}=0$	
Example 2:	$t \to t+it$, $0 \le t \le 1$	$\int_0^1 (t+it)^3\cdot(1+i)dt=\frac{1}{4}(1+i)^4$	
Example 3*:	$t \to t^2+it^2$, $0 \le t \le 1$	$\int_0^1 (t^2+it^2)^3(2t+2it)dt=\frac{1}{4}(1+i)^4$	
Example 4:	$t \to (1-t)+i(1-t)$, $0 \le t \le 1$	$\int_0^1 \{(1-t)+i(1-t)\}^3(-1-i)dt=-\frac{1}{4}(1+i)^4$	

N.B. (i) If we reparametrize the path of an arc by a proper reparametrization, we do not change the value of an integral along the arc. (This is the Reparametrization Theorem of Section 4.7.) We can therefore talk of the integral along a directed path.

(ii) If two arcs have the same path but have opposite directed paths, then the integrals have equal magnitude but opposite signs. (This is proved in Problem 4 of Section 4.6.)

(iii) Examples 2, 3, 4 illustrate that under certain conditions $\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a))$ where $[a,b]$ is the domain of γ. This is the Fundamental Theorem for Integrals along Arcs (Theorem 6 of Section 4.5) and is discussed in Television Programme 3.

5. Integration of a function along a contour (See Section 4.7.)

A contour is a string of directed paths of simple or simple-closed smooth arcs whose paths link together (see Section 4.7), so the above definition can easily be extended to them: to integrate along a contour we sum the integrals along its component arcs.

---

* Strictly speaking, this integral has not been defined, the given definition applying only to smooth arcs; however, the definition is sometimes extended to arcs for which the smoothness breaks down only by the derivative being zero at one or both end points, such as Example 3 here.

Self-Assessment Questions

1. Let γ be the arc defined by

$\gamma: t \to t + i(1 - t), \quad 0 \le t \le 1.$

Which of the following arcs are (a) reparametrizations of γ;

(b) proper reparametrizations of γ?

(i) $\gamma_1: t \to (1 - t) + it, \quad 0 \le t \le 1;$

(ii) $\gamma_2: t \to 2t + i(1 - 2t), \quad 0 \le t \le 2;$

2. Which of the following arcs are (a) closed; (b) simple;

(c) simple-closed; (d) smooth?

(i) $\gamma_1: t \to e^{it}, \qquad 0 \le t \le \pi.$

(ii) $\gamma_2: t \to e^{2it}, \qquad 0 \le t \le \pi.$

(iii) $\gamma_3: t \to t^2 + it^2, \qquad -1 \le t \le 1.$

3. Evaluate (a) by parametrization, (b) using the Fundamental Theorem for integrals along arcs

(i) $\int_\gamma z^2 dz$, where $\gamma: t \to e^{it}$, $0 \le t \le \pi$;

(ii) $\int_\gamma z\, dz$, where $\gamma: t \to t^2 + it$, $0 \le t \le 1$.

Solutions

1. (i) (a) but not (b);

(ii) (a) and (b);

2. (i) Simple and smooth;

(ii) Closed, simple-closed and smooth;

(iii) Closed.

3. (i) (a) $\gamma'(t) = ie^{it}$.

Therefore $\int_\gamma z^2 dz = \int_0^\pi e^{2it} \cdot ie^{it} dt$

$= \int_0^\pi ie^{3it} dt$

$= \frac{1}{3} e^{3it} \Big|_0^\pi$

$= -\frac{2}{3}.$

(b) $\int_\gamma z^2 dz = \frac{1}{3} z^3 \Big|_{e^{i0}}^{e^{i\pi}}$

$= -\frac{2}{3}$

(ii) (a) $\gamma'(t) = 2t + i$.

Therefore $\int_\gamma z\, dz = \int_0^1 (t^2 + it)(2t + i) dt$

$= \int_0^1 (2t^3 + 3it^2 - t) dt$

$= (\tfrac{1}{2}t^4 + it^3 - \tfrac{1}{2}t^2) \Big|_0^1$

$= i.$

(b) $\int_\gamma z\, dz = \frac{1}{2} z^2 \Big|_0^{1+i}$

$= i.$

M332 Television Programme 1

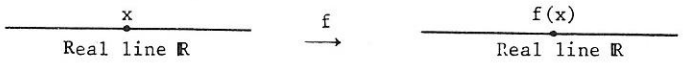
Continuous Functions

INTRODUCTION (to be read BEFORE WATCHING THE PROGRAMME)

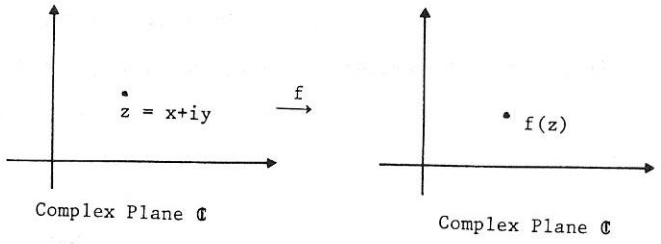
The programme introduces some of the fundamental definitions of complex analysis. It is NOT necessary to have read Unit 2 before watching the programme, but it would be well worth your while to read through this introduction to remind you of some basic ideas in complex analysis. Definitions in complex analysis are very often similar in appearance to those of real analysis, and it is important to remember the different interpretations of the notation.

1. Functions

Real analysis; $f: x \to f(x)$, $x \in \mathbb{R}$, $f(x) \in \mathbb{R}$.



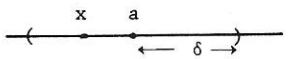
Complex analysis; $f: z \to f(z)$, $z \in \mathbb{C}$, $f(z) \in \mathbb{C}$.



2. Neighbourhoods

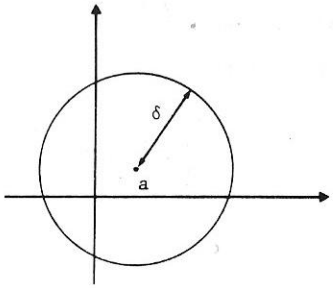
Real analysis; $|x-a| < \delta$ means

$x \in (a-\delta, a+\delta)$.



Complex analysis; $|z-a| < \delta$ means

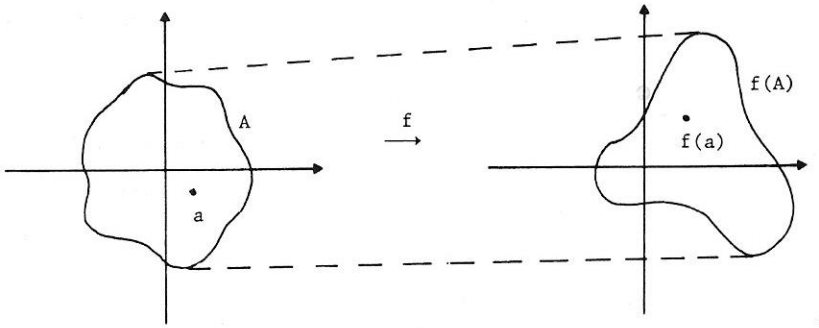
$z \in$ disc centred at a of radius δ.



3. Notation f(A)

f(A) is a set of points in the range of f, that is

$$f(A) = \{f(z) \in \mathbb{C} : z \in A\}.$$



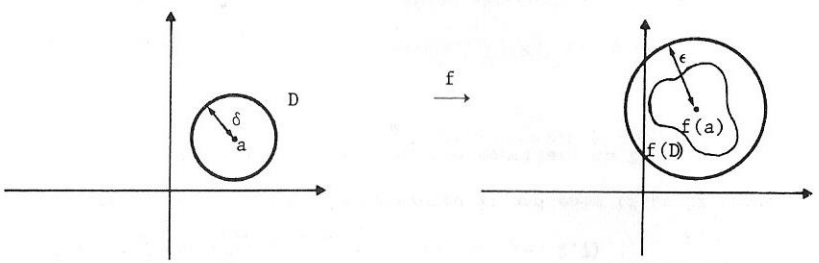
The notes below summarize the main points of the programme and are intended to save you from having to take any notes during the programme. We suggest that immediately after the programme you attempt the self assessment questions - which come after the notes - to help consolidate the ideas of the programme.

NOTES

1. Continuity of a function at a point (Section 2.3)

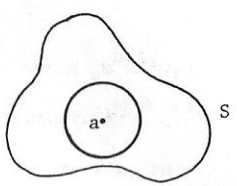
The function f is continuous at a if

$\lim_{z \to a} f(z) = f(a)$; (i.e. if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|z-a| < \delta$ implies $|f(z) - f(a)| < \varepsilon$).



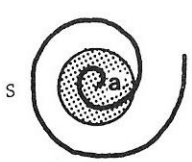
2. Open Set (Section 2.3)

S is an open set if every point of S is an interior point of S; (i.e. if for every $a \in S$, $\{z: |z-a| < \delta\} \subseteq S$ for some $\delta > 0$).



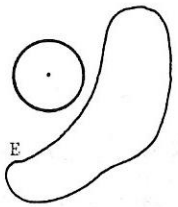
3. Cluster point (Section 2.5)

a is a cluster point of the set S if for all $\delta > 0$ there is a $z \in S$ such that $0 < |z-a| < \delta$.



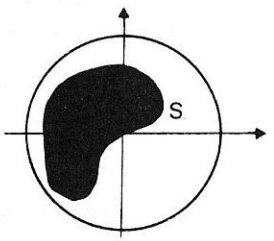
4. Closed set (Section 2.5)

A set E is called closed if every cluster point of E belongs to E.



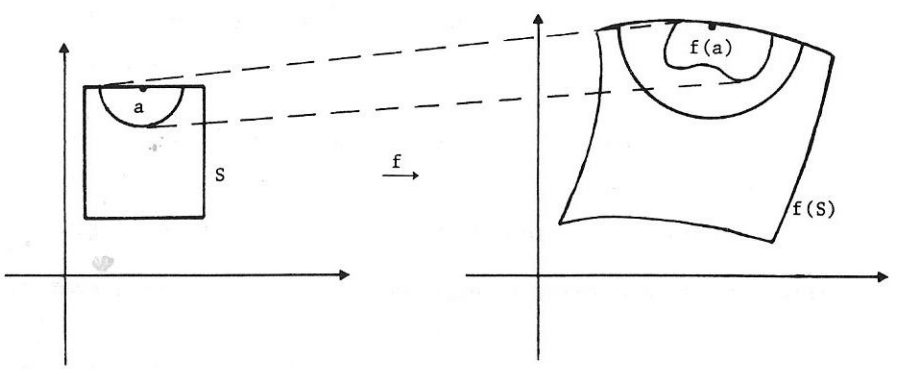
5. Bounded set (Section 1.5)

A set S is bounded if all the points of S lie in a closed disc with centre the origin; (i.e. if there is a real number k such that $|z| \leq k$ for all $z \in S$).



6. Continuity of a function on a set (Section 2.5)

The function f is continuous on the set S if for every $a \in S$, and for every $\varepsilon > 0$, there is some $\delta > 0$ such that for all z with $|z-a| < \delta$ and $z \in S$ then $|f(z) - f(a)| < \varepsilon$.



7. Connected Set (Section 2.7)

The open set G is disconnected if there exist non-empty open sets G_1 and G_2 such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \phi$. The open set G is connected if it is not disconnected.

8. Polygonally - connected set (Section 2.7)

The set G is polygonally - connected if for any two points z,z' in G there is a polygonal line $[z,z_1,\ldots,z_{n-1},z']$ contained in G. (Note that $[z,z_1,z_2,\ldots z_{n-1},z'] = [z,z_1] \cup [z_1,z_2] \cup \ldots \cup [z_{n-1},z']$, where $[z,z_1],\ldots,[z_{n-1},z']$ are line segments.)

9. Connected $\Leftrightarrow$ Polygonally-connected (Section 2.7)

An open set is polygonally-connected if and only if it is connected. (This is proved in Theorem 8 and the corollary to Theorem 7 of Unit 2).

10. Compact Set (Section 2.9)

A set S in $\mathbb{C}$ is called compact if no matter how an open disc D_a, centre a, is assigned to each point a of S, there are a finite number of elements $a_1,\ldots,a_n$ of S such that $S \subseteq D_{a_1} \cup \ldots \cup D_{a_n}$.

11. Some results on compact sets (Section 2.9)

(i) A set is compact if and only if it is closed and bounded. (This is equivalent to Theorem 10 and 11 of Unit 2.)

(ii) A continuous function f defined on a closed bounded set S has a bounded image f(S). ($f(S) = \{z : z = f(a)$ for some $a \in S\}$.) (This is equivalent to Theorem 12 of Unit 2.)

Self-Assessment Questions

1. Draw diagrams to illustrate the following concepts of complex analysis;

(i) continuity of a function at a point; (ii) open set;

(iii) cluster point; (iv) closed set;

(v) continuity of a function on a set; (vi) bounded set.

2. Define the following:

(i) connected set; (ii) polygonally-connected set;

(iii) compact set.

3. If S is an open set, which of the following statements about S are true?

(i) If S is connected, S is polygonally-connected.

(ii) If S is polygonally-connected, S is connected.

(iii) S is compact.

4. Let $S = \{z : z = x+iy,\ x = \frac{1}{n},\ n \in \mathbb{N},\ y = 0\}$. Use the definition of compactness to show that S is not compact.

5. Prove that if f is a continuous function defined on a closed bounded set S, then the set f(S) is bounded.

Solutions

1. (i) See Note 1. (ii) See Note 2. (iii) See Note 3.

(iv) See Note 4. (v) See Note 6. (vi) See Note 5.

2. (i) See Note 7 (i.e. an open set G is connected if there do not exist non-empty open sets G_1 and G_2 such that $G = G_1 \cup G_2$.

(ii) See Note 8. (iii) See Note 10.

3. (i) and (ii). (See Notes 9 and 11.)

4. We need to construct a set of discs centred at each point of the set S no finite number of which contain the whole of S.

Now the distance between 1/k and 1/(k+1) is $\frac{1}{k} - \frac{1}{k+1}$, that is $\frac{1}{k(k+1)}$.

If $a_k = 1/k$, let $D_{a_k} = \{z : |z-a_k| < 1/2k(k+1)\}$.

Then, if $j \neq k$, $a_j \notin D_{a_k}$, so there is no finite number of elements of S such that $S \subseteq D_{a_1} \cup \ldots \cup D_{a_n}$.

5. f is continuous, so for any $a \in S$ there is a $\delta_a > 0$ such that whenever $|z-a| < \delta_a$ and $z \in S$ then $|f(z) - f(a)| < 1$.

Let $D_a = \{z : z \in S \text{ and } |z-a| < \delta_a\}$

(i.e. given a disc centred at f(a) we can choose a disc D_a centred at a such that the image of D_a is entirely <u>within</u> the given disc centred at f(a)).

As a ranges over S the set of all the D_a covers S. (Figure 1)

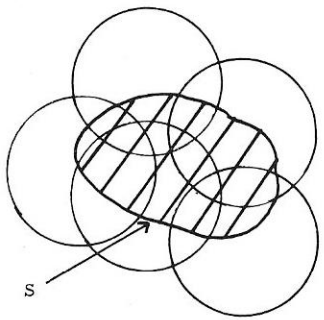


Figure 1

Corresponding to each $a \in S$ there is a disc D_a.

But S is compact and so a finite number of these discs covers S (Figure 2). Let us call these discs $D_{a_1}, \ldots, D_{a_n}$.

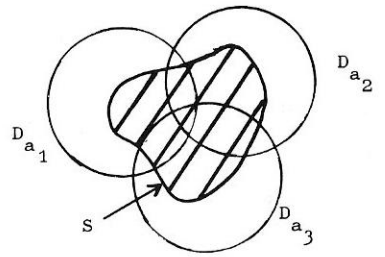


Figure 2

We have drawn this when n = 3, for clarity.

The image of each of these discs is contained in a disc of radius 1.

Let $k = \max\{|f(a_1)|, |f(a_2)|, \ldots, |f(a_n)|\} + 1$.

Then $|f(z)| \leq k$, for all $z \in S$ (Figure 3).

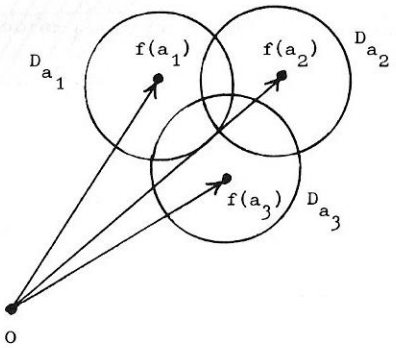


Figure 3

(Again n = 3.)

As each set $f(D_{a_i})$ lies in a disc with centre $f(a_i)$ and radius 1,

$z \in D_{a_i} \Rightarrow |f(z)| \leq |f(a_i)| + 1 \leq k$;

and hence

$z \in S \Rightarrow |f(z)| \leq k$.

That is, the set f(S) is bounded.

M332 Television Programme 2

Exp and Log

Television Broadcast Notes

INTRODUCTION (to be read BEFORE WATCHING THE PROGRAMME)

In this programme we discuss the generalization of the real, exponential and logarithm functions to complex functions. There are some important points to be aware of when we make such generalizations.

1. The new functions must have the same basic properties as the real functions exp and log. You may recall that these are;

Properties of REAL-VALUED Functions

(a) $\exp(x+y) = \exp x \cdot \exp y$, for all $x,y \in \mathbb{R}$,

(b) $\exp'(x) = \exp x$, for all $x \in \mathbb{R}$,

(c) log is the inverse function of exp on a suitable domain.

2. Since $\mathbb{R}$ is a subset of the complex plane $\mathbb{C}$ our new functions must yield the old functions when we consider just this subset.

If you have some time to spare before watching the programme you may like to try the following preparatory questions.

Preparatory Questions

1. Find the modulus of each of the following complex numbers (where cos, sin and f are all real functions)

(i) $\cos y + i \sin y$

(ii) $f(x) \cdot (\cos y + i \sin y)$, where $f(x) > 0$, for all $x \in \mathbb{R}$.

2. Show, on the complex plane, the point $e(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$.

Solutions

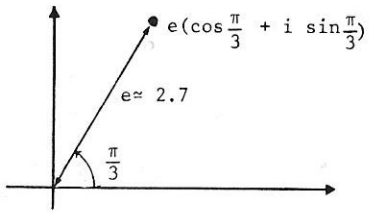
1. (i) $|\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = \sqrt{1} = 1.$

(ii) $|f(x) \cdot (\cos y + i \sin y)| = |f(x)| \, |\cos y + i \sin y|$

$= |f(x)|$

$= f(x)$, since $f(x) > 0$.

2. The point $e(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ is shown on the diagram below.



The notes below summarize the main points of the programme and are intended to save you from having to take any notes during the programme. We suggest that immediately after the programme you attempt the self assessment questions - which come after the Notes - to help consolidate the ideas of the programme.

NOTES

1. Definition of exp

$\exp z = e^x(\cos y + i \sin y)$, where $z = x + iy$, x and y are real, and e^x, sin and cos are the familiar functions of real analysis

(Some motivation for this definition is given in Section 3.4).

2. Addition formula for exp

$\exp(z_1+z_2) = \exp z_1 \cdot \exp z_2$

3. Differentiation formula for exp

$\exp' = \exp$.

This is Theorem 9 of Section 3.4.

4. Modulus and argument of exp

$|\exp z| = e^x$. (This is a particular case of P Q 1(ii) above.)

From the definitions, an argument of exp z is y.

5. Solution of exp z = w, $w \neq 0$

The function exp maps the complex plane onto the complex plane with the origin removed, so we should be able to solve for z the equation.

$$\exp z = w, \quad w \neq 0.$$

Taking the modulus of both sides, we get

$$e^x = |w|,$$

and so, taking log (the real logarithm) of both sides,

$$x = \log |w|.$$

We know that, for some θ,

$$w = |w| (\cos \theta + i \sin \theta).$$

Also, $\exp z = e^x (\cos y + i \sin y)$.

Equating real and imaginary parts,

$$\cos \theta = \cos y \text{ and } \sin \theta = \sin y,$$

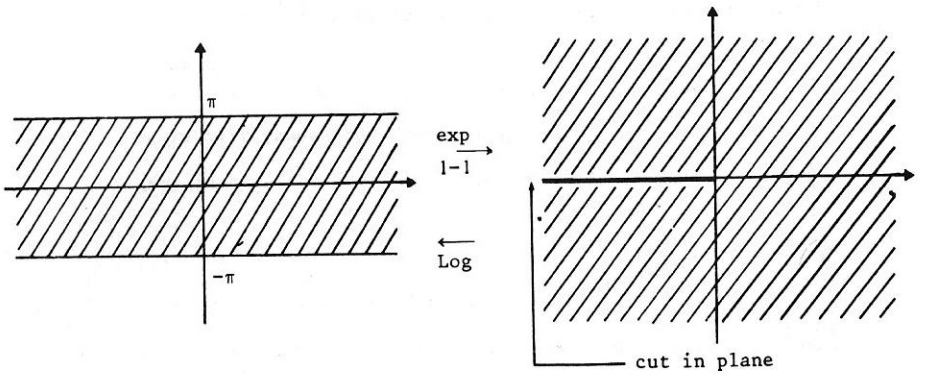
so $y = \theta + 2n \cdot \pi$ for some integer n.

Therefore $z = \log |w| + i\theta + 2n\pi i$, for some integer n.

M332 TV 2

6. Definition of Log (the principal branch of the logarithm)

In order to define an inverse of exp, we need to restrict the domain of exp so as to make it one-one. We define Log to be the inverse of the one-one function exp with domain $\{z: -\pi < \text{Im } z < \pi\}$.



$\exp(\text{Log } z) = z$

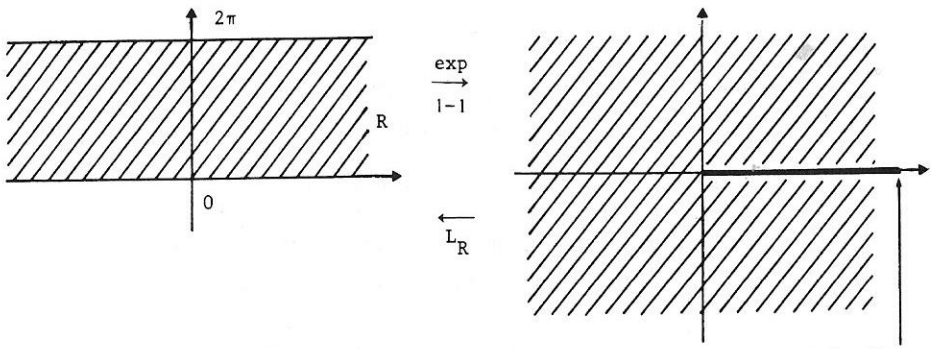
$\text{Log } z = \log |z| + i \text{Arg } z$

7. Definitions of L_R (the branch of the logarithm associated with R)

If we want a logarithm of -1 (for example), we need to choose a fundamental region for exp (see Note 8 below) which includes -1 in its image under exp. For example, we can take the region

$$R = \{z : 0 < \text{Im } z < 2\pi\}.$$

So we define L_R to be the inverse of the 1-1 function exp with domain R.



$\exp(L_R(z)) = z$

$L_R(z) = \log|z| + i\theta$

(θ is the argument of z such that $0 < \theta < 2\pi$.)

8. Fundamental regions for exp

A fundamental region for exp is a region R such that exp is one-one on R, but on no region S properly containing R (i.e. such that $R \subset S$, but $R \neq S$) (Section 3.6). There are infinitely many fundamental regions for exp. We can always choose such a region so that any particular point (except zero) is included in its image.

M332 TV 2

Self-Assessment Questions

1. Show on a diagram the points $2 + \frac{i\pi}{3}$ and $\exp(2 + \frac{i\pi}{3})$

2. (a) If $S = \{z : \exp z = w\}$ and $z_1, z_2 \in S$, what can you say about $z_1 - z_2$?

   (b) Is exp periodic? If so, what is a period?

3. Sketch S and exp S in each of the following cases:

   (i) $S = \{c + iy : c \text{ a real constant}, \frac{\pi}{4} \leq y \leq \frac{\pi}{2}\}$.

   (ii) $S = \{x + ic : c \text{ a real constant}, 0 \leq x \leq \log(\frac{3}{2})\}$.

   (iii) $S = \{z : \log 1 \leq \text{Re } z \leq \log(\frac{3}{2}), \frac{\pi}{4} \leq \text{Im } z \leq \frac{\pi}{2}\}$.

4. Find (where possible) Log z and $L_R(z)$, for $R = \{z : 0 < \text{Im } z < 2\pi\}$, in each of the following cases.

   (i) $z = -1 + i$; (ii) $z = -1 - i$; (iii) $z = 1$; (iv) $z = -1$.

5. What is the relationship between Log z and $L_R(z)$, where R is defined as in Self-Assessment Question 4?

6. Prove that

   $$\text{Log}(z_1 z_2) = \text{Log } z_1 + \text{Log } z_2 + 2n\pi i \text{ for some integer } n.$$

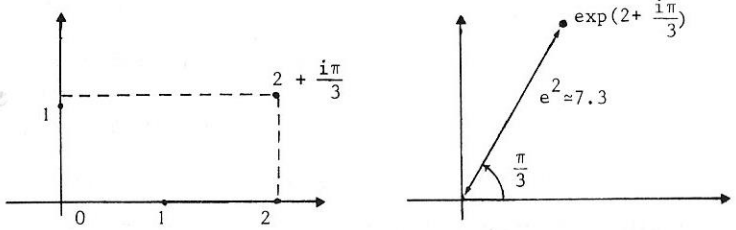
7. Is it true that

   $$\text{Log}(z_1 z_2) = \text{Log } z_1 + \text{Log } z_2?$$

   Give a proof or a counter-example.

Solutions

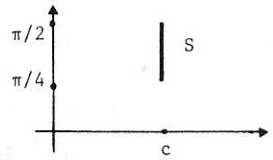
1.

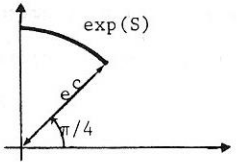


2. (a) $z_1 - z_2 = 2n\pi i$ for some integer n.

(b) Yes. $2n\pi i$ for any integer n.

3. (i)





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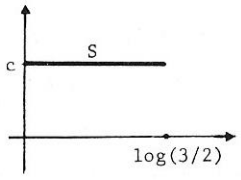
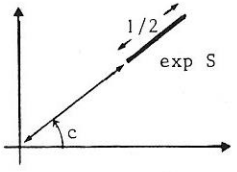
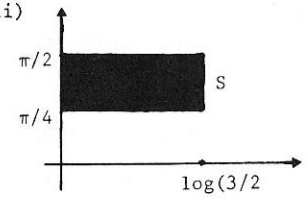
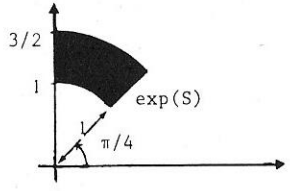




(iii)





4. (i) $\text{Log}(-1+i) = \log\sqrt{2} + i\frac{3\pi}{4}$; $L_R(-1+i) = \log\sqrt{2} + i\frac{3\pi}{4}$;

(ii) $\text{Log}(-1-i) = \log\sqrt{2} - i\frac{3\pi}{4}$; $L_R(-1-i) = \log\sqrt{2} + i\frac{5\pi}{4}$;

(iii) $\text{Log}(1) = \log 1 = 0$; $L_R(1)$ is undefined;

(iv) $\text{Log}(-1)$ is undefined; $L_R(-1) = i\pi$.

5. If Im z > 0; $L_R(z) = \text{Log } z$.

If Im z < 0; $L_R(z) = \text{Log } z + 2\pi i$.

If Im z = 0; no relationship can exist between Log(z) and $L_R(z)$ because if Re z < 0, then, Log z does not exist but $L_R(z)$ does, whereas if Re z > 0, then $L_R(z)$ does not exist but Log z does, and if Re z = 0, neither exists.

6. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, where $\theta_1 = \text{Arg } z_1$, $\theta_2 = \text{Arg } z_2$.

Then $\text{Log } z_1 = \log r_1 + i\theta_1$, $\text{Log } z_2 = \log r_2 + i\theta_2$

and $\text{Log } z_1 + \text{Log } z_2 = \log r_1 + \log r_2 + i(\theta_1+\theta_2)$

$= \log r_1r_2 + i(\theta_1+\theta_2)$.

$\text{Log}(z_1z_2) = \log r_1r_2 + i\phi$, where $\phi = \theta_1+\theta_2 + 2n\pi$, the integer n being chosen so that $-\pi < \phi < \pi$.*

$= \text{Log } z_1 + \text{Log } z_2 + 2n\pi i$.

*N.B. This is not possible if $z_1z_2 = -x$, for some real x, because then z_1z_2 is not in the domain of Log.

7. No. For example,

let $z_1 = -1+i$, $z_2 = -1+i$, so $z_1z_2 = -2i$.

$\text{Log } z_1 = \text{Log } z_2 = \log\sqrt{2} + i\frac{3\pi}{4}$

so $\text{Log } z_1 + \text{Log } z_2 = 2\log\sqrt{2} + i\frac{3\pi}{2}$

$= \log 2 + i\frac{3\pi}{2}$.

But $\text{Log}(z_1z_2) = \log 2 - i\frac{\pi}{2}$.

So $\text{Log}(z_1z_2) \neq \text{Log } z_1 + \text{Log } z_2$.

M332 Radio Programme 3

Radio Broadcast Notes

Taylor Series

INTRODUCTION (to be read BEFORE LISTENING TO THE PROGRAMME)

This programme is intended to help you with some of the techniques that are useful in calculating Taylor coefficients. It complements the short piece in Unit 6 at the beginning of Section 6, which you might like to look at after the programme. As a consequence of the Uniqueness Theorem, the calculation of Taylor series for a complex variable is very similar to that for a real variable. You might like to spend a minute or two before the programme reminding yourself of some relevant facts by working through the following questions.

Preparatory Questions

1. State the Uniqueness Theorem.

2. What are the Taylor series centred at the origin for the following real functions? Give the values of x for which each is convergent.
   (i) $(1+x)^{-1}$; (ii) sin x; (iii) cos x; (iv) exp x.

Solutions

1. Let f and g be analytic on a region R, S be a subset of R with a cluster point in R, and f = g on S. Then f = g on R.
   (Theorem 16 of Section 6.7).

2. See the first part of Note 1 below.

Throughout the programme, we concentrate on Taylor series centred at the origin. The principles involved apply to all Taylor series; we have centred them at the origin merely to make the equations simpler to read.

As with the acquisition of most techniques, there can be no substitute for practice; so your work on the self assessment questions immediately after listening to the programme will be particularly valuable.

NOTES (to be followed DURING THE PROGRAMME)

1. Some important Taylor series (about 0)

Real Series

$$(1+x)^{-1} = 1-x+x^2-x^3+\ldots, \qquad |x| < .$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots, \qquad x \in \mathbb{R}.$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots, \qquad x \in \mathbb{R}.$$

$$\exp x = 1+x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \qquad x \in \mathbb{R}.$$

Complex Series

$$(1+z)^{-1} = 1-z+z^2-z^3+\ldots, \qquad |z| < 1.$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots, \qquad z \in \mathbb{C}.$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots, \qquad z \in \mathbb{C}.$$

$$\exp z = 1+z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots, \qquad z \in \mathbb{C}.$$

2. Combinations of Taylor series

Example 1 Find the Taylor series (about 0) for $z \to \frac{1}{(1-z)(1+2z)}$.

Method 1 (multiplying the appropriate series)

$$\frac{1}{(1-z)(1+2z)} = \frac{1}{1+2z} \cdot \frac{1}{1-z}.$$

(1) $$\frac{1}{1+2z} = 1 - 2z + (2z)^2 - \ldots, \quad |2z| < 1;$$

(2) $$\frac{1}{1-z} = 1 + z + z^2 + \ldots, \quad |z| < 1;$$

(3) $$\frac{1}{(1-z)(1+2z)} = (1-2z+(2z)^2-\ldots)\cdot(1+z+z^2+\ldots),$$

$$|2z| < 1 \text{ and } |z| < 1,$$

$$= 1-z+(4-2+1)z^2-\ldots, \quad |z| < \tfrac{1}{2},$$

$$= 1 - z + 3z^2-\ldots .$$

Method 2 (adding appropriate series)

$$\frac{1}{(1-z)(1+2z)} = \frac{1}{3}\left(\frac{1}{1-z} + \frac{2}{1+2z}\right)$$

$$= \frac{1}{3}\left[(1+z+z^2+\ldots) + 2(1-2z+(2z)^2-\ldots)\right],$$

$$|2z| < 1 \text{ and } |z| < 1,$$

$$= 1 - z + 3z^2-\ldots .$$

Example 2 Find the Taylor expansion (about 0) for $\frac{1}{1-z-z^2}$.

Solution

Let $\frac{1}{1-z-z^2} = \sum_{n=0}^{\infty} a_n z^n$,

then $1 = (a_0+a_1z+\ldots)(1-z-z^2)$ and, by the uniqueness of Taylor series, comparing coefficients of

(i) z^0: $a_0 = 1$;

(ii) z^1: $a_1-a_0 = 0$, so that $a_1 = 1$;

(iii) z^n for $n>1$: $a_n-a_{n-1} - a_{n-2} = 0$, so that $a_n = a_{n-1} + a_{n-2}$.

(This recurrence relation defines the Fibonacci Numbers which crop up in many places in mathematics.)

Suppose $1-z-z^2 = (\alpha-z)(z-\beta)$.

Then $$\frac{1}{1-z-z^2} = \frac{1}{\alpha-\beta}\left[\frac{1}{\alpha-z} - \frac{1}{\beta-z}\right]$$

$$= \frac{1}{\alpha-\beta}\left[\frac{1}{\alpha(1-z/\alpha)} - \frac{1}{\beta(1-z/\beta)}\right]$$

$$= \frac{1}{\alpha-\beta}\left[\frac{1}{\alpha}\sum_{n=0}^{\infty}\left(\frac{z}{\alpha}\right)^n - \frac{1}{\beta}\sum_{n=0}^{\infty}\left(\frac{z}{\beta}\right)^n\right], \text{ for}$$

$$\left|\frac{z}{\alpha}\right| < 1 \text{ and } \left|\frac{z}{\beta}\right| < 1,$$

(from the standard expansion for $1/(1-z)$).

So $a_n = \frac{1}{\alpha-\beta}\left[\frac{1}{\alpha^{n+1}} - \frac{1}{\beta^{n+1}}\right]$.

α and β are the solutions of $1 - z - z^2 = 0$,

so $\alpha = \frac{1}{2}(\sqrt{5}-1)$ and $\beta = -\frac{1}{2}(\sqrt{5}+1)$.

This gives

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{2}{\sqrt{5}-1}\right)^{n+1} - \left(\frac{-2}{\sqrt{5}+1}\right)^{n+1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

For convergence we require $\left|\frac{z}{\alpha}\right| < 1$ and $\left|\frac{z}{\beta}\right| < 1$, so the

3. Use of Taylor series to find the order of a zero of an analytic function

α is a zero of the analytic function f if $f(\alpha) = 0$.

If α is a zero of order n, then there is an analytic function g such that $f(z) = (z-\alpha)^n \cdot g(z)$, and $g(\alpha) \neq 0$.

(f,g are analytic on some region R, where $\alpha \in R$.)

Example 3 sin has a zero at 0. What is the order of this zero?

Solution

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots, \quad z \in \mathbb{C}.$$

$$= z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots\right)$$

$= z \cdot g(z)$, where g is entire and $g(0) \neq 0$.

So sin has a zero of order 1 at 0.

Self-Assessment Questions (to be attempted AFTER THE PROGRAMME)

Questions

1. (a) Express $1/[(1-z)(1+2z)]$ in partial fractions.

   (b) Hence, using the lines labelled (1) and (2) in Example 1, find the first three terms of the Taylor series about 0 for $z \to 1/[(1-z)(1+2z)]$.

2. Use both methods of Example 1 to find the first three terms of the Taylor series about 0 for $z \to 1/[(1+z)(1-3z)]$.

3.(i) Use the method of Example 2 to find a recurrence relation for the coefficients of the Taylor series about 0 for $z \longrightarrow 1/(z^2+z+1)$.

(ii) Adapt the method to find a recurrence relation for the coefficients of the Taylor series about 1 for the same function.

4. Use the appropriate Taylor series to find the order of the zero of sinh at 0.

Solutions

1. (a) Let $\frac{1}{(1-z)(1+2z)} = \frac{A}{1-z} + \frac{B}{1+2z}$,

so $1 = A(1+2z) + B(1-z)$, $z \neq 1$, $z \neq -\frac{1}{2}$.

Equating coefficients gives $B = 2/3$, $A = 1/3$, respectively.

Therefore $\frac{1}{(1-z)(1+2z)} = \frac{1}{3}\left[\frac{1}{1-z} + \frac{2}{1+2z}\right]$, as stated in Note 2.

(b) Using the series labelled (1) and (2) in Note 2, we get

$$\frac{1}{(1-z)(1+2z)} = \frac{1}{3}\left[1+z+z^2+\ldots+2\left[1-2z+(2z)^2-\ldots\right]\right], \quad |z| < \tfrac{1}{2}.$$

$$= \frac{1}{3}\left[3-3z+9z^2-\ldots\right]$$

$= 1-z+3z^2-\ldots$, which agrees with the result of Method 1.

2. $\frac{1}{1+z} = 1-z+z^2-z^3+\ldots, \quad |z| < 1$

$\frac{1}{1-3z} = 1+3z+(3z)^2+(3z)^3+\ldots, \quad |3z| < 1$

Method 1

$$\frac{1}{(1+z)(1-3z)} = (1-z+z^2-\ldots)\cdot(1+3z+(3z)^2+\ldots), \quad |z| < \frac{1}{3}$$

$$= 1+2z+z^2(1-3+9)+\ldots$$

$$= 1+2z+7z^2+\ldots .$$

Method 2

Let $\frac{1}{(1+z)(1-3z)} = \frac{A}{1+z} + \frac{B}{1-3z}$,

so $1 = A(1-3z) + B(1+z)$.

This gives $A = 1/4$, $B = 3/4$.

Therefore $\frac{1}{(1+z)(1-3z)} = \frac{1}{4}\left[\frac{1}{1+z} + \frac{3}{1-3z}\right]$

$$= \frac{1}{4}\left[(1-z+z^2-\ldots) + 3\left(1+3z+(3z)^2\ldots\right)\right], \quad |z| < \frac{1}{3}.$$

$$= \frac{1}{4}(4+8z+28z^2+\ldots)$$

$$= 1+2z+7z^2+\ldots .$$

3.(i) Let $\frac{1}{z^2+z+1} = \sum_{n=0}^{\infty} a_n z^n$;

then $1 = (a_0+a_1z+a_2z^2+\ldots)(z^2+z+1)$ and, by the uniqueness of Taylor series, comparing the coefficients of

(i) z^0: $a_0 = 1$;

(ii) z^1: $a_0+a_1 = 0$, so that $a_1 = -1$;

(iii) z^n for $n>1$: $a_n+a_{n-1}+a_{n-2} = 0$, so that $a_n = -a_{n-1}-a_{n-2}$

which is the required recurrence relation.

(ii) Suppose $\frac{1}{z^2+z+1} = \sum_{n=0}^{\infty} b_n(z-1)^n$;

then, using the identity $z^2+z+1 = (z-1)^2+3(z-1)+3$ to express z^2+z+1 as a polynomial in $(z-1)$, we have

$$1 = \left(b_0+b_1(z-1)+b_2(z-1)^2+\ldots\right)\left((z-1)^2+3(z-1)+3\right).$$

By the uniqueness of Taylor series, comparing the coefficients of

(i) $(z-1)^0$: $3b_0 = 1$, so that $b_0 = \frac{1}{3}$;

(ii) $(z-1)^1$: $3b_1+3b_0 = 0$, so that $b_1 = -\frac{1}{3}$;

(iii) $(z-1)^n$ for $n>1$: $3b_n+3b_{n-1}+b_{n-2} = 0$, so that $b_n = -b_{n-1}-\frac{1}{3}b_{n-2}$.

4. $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots$

$= z(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots)$

$= zg(z)$ where g is entire and $g(0) \neq 0$.

So sinh has a zero of order 1 at 0.

Television Broadcast Notes

M332 Television Programme 3

Complex Integration

INTRODUCTION (to be read BEFORE WATCHING THE PROGRAMME)

Before watching this programme you are expected to know what is meant by integrals along an arc and a contour and to know how to evaluate such integrals by parametrization. In the programme we look at various techniques for evaluating integrals by examining the three integrals $\int_C \frac{dz}{z}$, $\int_C \frac{dz}{z+2}$, $\int_C \frac{dz}{z-\frac{1}{2}}$, where C in each case denotes the unit circle. To prepare you for the programme, we suggest that you spend a few minutes on the following preparatory questions.

Preparatory Questions

1. State the Fundamental Theorem for contour integrals.
2. Evaluate (i) $\int_\gamma z^2 dz$, where $\gamma(t) = t(1+i)$, $t \in [1,4]$;

   (ii) $\int_\gamma \frac{1}{z} dz$, where $\gamma(t) = e^{it}$, $t \in [0,2\pi]$.
3. State Cauchy's Formula.
4. State Cauchy's Theorem for star regions.

Solutions

1. Let f be analytic on a region R, and Γ be a contour in R. Suppose also that f' is continuous on R. Then $\int_\Gamma f'(z)dz = f(\beta) - f(\alpha)$, where α is the initial and β the final point of Γ. Furthermore, if Γ is closed, then $\int_\Gamma f'(z)dz = 0$ (Theorem 8 of Unit 4).

2. (i) $\int_\gamma z^2 dz = \int_1^4 [t(1+i)]^2 (1+i)\, dt = (1+i)^3 \int_1^4 t^2 dt = 21\ (1+i)^3$.

(ii) $\int_\gamma \frac{1}{z}\, dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\, dt = \int_0^{2\pi} i\, dt = 2\pi i$.

3. Let C be the circle $|z - \alpha| = r$, with α inside D. Let f be analytic on some region R containing DUC. Then, for any z in D,
$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta$ (Theorem 4 of Unit 5).

4. Let f be a function analytic on a star region R, and let Γ be any closed contour in R. Then $\int_\Gamma f = 0$ (Theorem 3 of Unit 5).

The notes below summarize the main points of the programme and are intended to save you from having to take any notes during the programme. We suggest that immediately after the programme you attempt the self assessment questions - which come after the notes - to help consolidate the ideas of the programme in your mind.

NOTES

1. Parametrization of the contour

Example 1: $\int_C \frac{1}{z}\, dz$, where C is the unit circle.

We can parametrize the unit circle by $t \to e^{it}$, $0 \le t \le 2\pi$.

So $\int_C \frac{1}{z}\, dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\, dt = \int_0^{2\pi} i\, dt = 2\pi i$.

N.B. The parametrization was chosen to make the point go round the unit circle precisely once and in an anti-clockwise direction, according to the normal convention for integrating around simple-closed contours. Any

M332 TV 3

parametrization satisfying these conditions will lead to the same value for the integral. Parametrizations which travel the path in a different direction, or more than once, will, in general, lead to different values for the integral.

2. Using the Fundamental Theorem for integrals along arcs

Example 2 : $\int_C \frac{1}{z+2}\, dz$, C being the unit circle.

If we use the parametrization $t \to e^{it}$, $0 \le t \le 2\pi$ the integral becomes $\int_0^{2\pi} \frac{ie^{it}}{e^{it}+2}\, dt$, which is awkward to simplify.

But we know that in real analysis $\int \frac{dx}{x+2} = \log(x+2)$ and we can use the Fundamental Theorem (see Preparatory Question 1 above) to obtain a similar result here. The function $f': z \to \frac{1}{z+2}$ is continuous on regions containing C such as $\{z: \text{Re } z > -2\}$; and $f: z \to \text{Log}(z+2)$ is analytic on such a region. (The necessary cut in the plane goes from -2 to $-\infty$, which is well clear of C.)

So $\int_C \frac{1}{z+2}\, dz = \text{Log}(z+2)\big|_C = 0$, because C is a closed contour.

(The notation $f(z)\big|_C$, is used to indicate the appropriate change in f(z) as z travels along C.)

N.B. (i) This method cannot be used for $\int_C \frac{1}{z}\, dz$ because $z \to \text{Log } z$ is defined only in the cut plane $\{z: z \neq x \text{ for some real } x \le 0\}$, so C is not contained in its domain. (Taking a different branch of the logarithm would not have helped, since any cut from 0 to infinity must intersect C.)

(ii) The above method will apply to a large class of closed contours in the same cut plane, including more complicated contours such as this,

as will be shown later in the course.

3. Cauchy's Formula

Example 3: $\int_C \frac{1}{\zeta-\frac{1}{2}}\,d\zeta$, C being the unit circle.

Note that we have changed here from using z to ζ for the variable, to correspond with the notation of Unit 5.

As in Example 2, parametrization by $t \to e^{it}$, $0 \le t \le 2\pi$, leads to complications. Also $\mathrm{Log}(\zeta-\frac{1}{2})$ is not defined (let alone differentiable) at $\zeta = -1$, so the conditions for the Fundamental Theorem are not satisfied. (And, once again, taking a different branch of the logarithm does not help.) Upon putting $z = \frac{1}{2}$ and $f(z) = 1$ for all z, the conditions for Cauchy's Formula (see Preparatory Question 3 above) are satisfied and

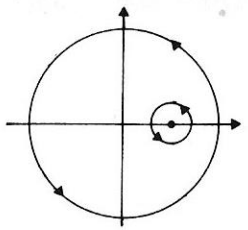
$$\int_C \frac{f(\zeta)}{\zeta-z}\,d\zeta = 2\pi i \cdot f(z)$$

becomes

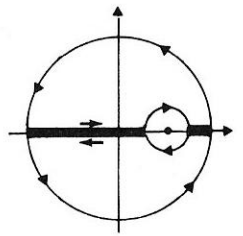
$$\int_C \frac{1}{\zeta-\frac{1}{2}}\,d\zeta = 2\pi i.$$

One can verify this particular application of Cauchy's Formula using the same method as is used for the general formula in Section 5.5:

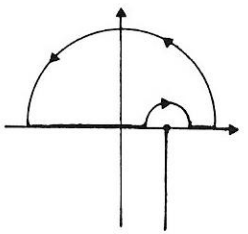
We shall show that the integral round the unit circle is the same as the integral taken in the same direction round a small circle, centre $\frac{1}{2}$. First we put arrows anticlockwise round the big circle and clockwise round the small circle.



We then replace the two circles by the two closed contours shown in the figure.



We first consider the top contour. The integral we are dealing with involves a logarithm, so we put a cut in the plane avoiding the contour as shown in the figure.



The conditions for Cauchy's Theorem for star regions (see Preparatory Question 4) are satisfied and so the integral around this contour is zero. Similarly we can prove that the integral around the bottom one is zero.

Denoting the integral around the outer circle anticlockwise by I and the inner circle clockwise by J, we have upon combining the above results that

$$I + J = 0, \text{ so that } I = -J \text{ as required.}$$

But we can find the integral around the inner circle by parametrization (see Self-Assessment Question 1 below). This gives $J = -2\pi i$, so that $I = 2\pi i$.

4. Geometric Point of View (using the geometric notion of winding number)

The winding number, Wnd(C,α), is defined by

$$\int_C \frac{1}{\zeta-\alpha}\,d\zeta = 2\pi i \text{ Wnd } (C,\alpha),$$

where Wnd (C,α) is the number of times the smooth arc C winds round the point α. (This result is stated in Unit 4 - see Section 4.6 - but is not proved until later in the course. But, in any case, the intuitive meaning is clear.)

By choosing suitable values for α in the above formula you can check that the values we obtained for $\int_C \frac{1}{z}\,dz$, $\int_C \frac{1}{z+2}\,dz$, $\int_C \frac{1}{\zeta-\frac{1}{2}}\,d\zeta$ are correct.

Self-Assessment Questions

Questions

1. Evaluate the following integral by using a suitable parametrization.

$$\int_{C_1} \frac{1}{\zeta-\frac{1}{2}}\,d\zeta, \text{ where } C_1 \text{ is the circle } \{z:|z-\tfrac{1}{2}| = \mathbf{r}\} \quad (r < \tfrac{1}{2}).$$

2. Which of the following integrals can be evaluated using the Fundamental Theorem?

(i) $\int_{C_1} \frac{1}{\zeta-\frac{3}{8}}\,d\zeta$, where C_1 is the circle $\{z:|z-\frac{1}{2}| = \frac{1}{4}\}$

(ii) $\int_{C_2} \frac{1}{z+2}\,dz$, where C_2 is the circle $\{z:|z-i| = 1\}$

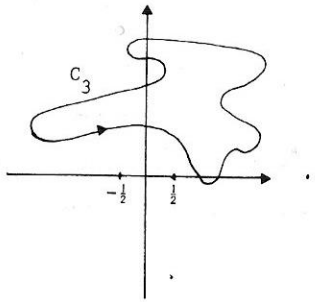
3. Evaluate the following integrals, using the identity

$$\frac{1}{(z-\frac{1}{2})(z+\frac{1}{2})} = \frac{1}{z-\frac{1}{2}} - \frac{1}{z+\frac{1}{2}}.$$

(i) $\int_C \frac{1}{(z-\frac{1}{2})(z+\frac{1}{2})}\,dz$, where C is the unit circle;

(ii) $\int_{C_1} \frac{1}{(z-\frac{1}{2})(z+\frac{1}{2})}\,dz$, where C_1 is the circle $\{z:|z-\frac{1}{2}| = \frac{1}{4}\}$;

(iii) $\int_{C_3} \frac{1}{(z-\frac{1}{2})(z+\frac{1}{2})}\,dz$, where C_3 is the contour shown below.



Solutions

1. (i) Let $\gamma: t \to \frac{1}{2} + re^{it}$, $0 \le t \le 2\pi$.

Then $\int_{C_1} \frac{1}{\zeta-\frac{1}{2}}\,d\zeta = \int_0^{2\pi} \frac{rie^{it}}{re^{it}}\,dt = 2\pi i.$

2. (i) Log $(\zeta - 3/8)$ is not defined (let alone differentiable) at $\frac{1}{4}$, so the Fundamental Theorem cannot be used here.

(ii) -2 is well clear of the contour C_2, so the Fundamental Theorem can be used here.

3. (i) $\int_C \frac{1}{(z-\frac{1}{2})(z+\frac{1}{2})}\,dz = \int_C \frac{1}{z-\frac{1}{2}}\,dz - \int_C \frac{1}{z+\frac{1}{2}}\,dz$

$= 2\pi i - 2\pi i$

$= 0.$

(ii) $\int_{C_1} \frac{1}{(z-\frac{1}{2})(z+\frac{1}{2})}\,dz = \int_{C_1} \frac{1}{z-\frac{1}{2}}\,dz - \int_{C_1} \frac{1}{z+\frac{1}{2}}\,dz$

$= 2\pi i - 0$

$= 2\pi i.$

(iii) $\int_{C_3} \frac{1}{(z-\frac{1}{2})(z+\frac{1}{2})}\,dz = \int_{C_3} \frac{1}{z-\frac{1}{2}}\,dz - \int_{C_3} \frac{1}{z+\frac{1}{2}}\,dz$

$= \text{Log}\,(z-\frac{1}{2})\big|_{C_3} - \text{Log}\,(z+\frac{1}{2})\big|_{C_3}$ (by the Fundamental Theorem)

$= 0 - 0$ (from above, or using winding numbers)

$= 0.$

M332 Television Programme 4

Taylor Series

INTRODUCTION (to be read BEFORE WATCHING THE PROGRAMME)

This programme is intended to help you find your way through the theorems in Unit 6. It shows you how they link up and to where they are leading. In particular, we ask three questions. Upon what sort of sets do power series converge? Are the functions which they define analytic? Can real power series be generalized to complex power series in order to define complex analytic functions?

We refer to various results of Unit 5 including Morera's Theorem (Theorem 9 of Section 5.7) that if f is a function continuous on a region R such that for any triangular contour Δ in R with inside contained in R, $\int_\Delta f = 0$, then f has an analytic antiderivative on R. Obviously this in turn implies that f is analytic on R. Some of the other relevant results from Unit 5 are listed in Note 1 below.

NOTES

1. Some properties of analytic functions

(i) Cauchy's Theorem for Star Regions (Theorem 3 of Section 5.3): Let f be a function analytic on a star region R and let Γ be any closed contour in R. Then $\int_\Gamma f = 0$.

(ii) The Analyticity of Derivatives (Theorem 7 of Section 5.5):
Let f be analytic on a region R. Then f' is analytic on R.
N.B. This implies that an analytic function is infinitely differentiable.

(iii) Liouville's Theorem (Theorem 10 of Section 5.7):
If f is a bounded, entire function, then f is constant.

2. How to define some analytic complex functions

The real function $x \to |x|$, $x \in \mathbb{R} - \{0\}$ is differentiable. However, the complex function $z \to |z|$, $z \in \mathbb{C} - \{0\}$ is not analytic anywhere. (See Theorem 6 of Unit 3.) So we cannot extend just any differentiable real function and be sure to obtain an analytic complex function.

3. Power series

A power series centred at α is a series of the form

$$a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \ldots + a_n(z-\alpha)^n + \ldots, \quad a_i \in \mathbb{C}.$$

For most of the programme, we let $\alpha = 0$ to simplify the writing. The proofs can be generalized easily to $\alpha \neq 0$. If a function f is specified by $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$, the domain of f is the set of values of z for which this series converges.

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4. The domain of f is a disc centred at the origin (Theorem 4 of Section 6.3)

One can remember the proof of this theorem as follows. Call a circle centre the origin inside which the series converges a C-circle, and one outside which it diverges a D-circle. Circles inside a C-circle must be C-circles and outside a D-circle must be D-circles. Each point of the plane must lie on one of these circles and there is only one circle, D_L, say, common to both sets of circles.

D_L is called the circle of convergence: the series converges at all points inside it and diverges outside. (We cannot say what happens for points on D_L.) So the domain of f is a disc with boundary D_L.

N.B. The corresponding real series converges on the interval which is the diameter of the circle of convergence.

5. A function f is analytic on D, the inside of its circle of convergence (Theorem 7 of Section 6.3)

The proof is on page 120 of Unit 6.

6. Every function analytic on an open disc D can be expressed as a Taylor series on D (Taylor's Theorem, Theorem 11 of Section 6.5)

If f is the function, if α is the centre of D, and if z is any point in D, then

$$f(z) = \sum_{r=0}^{\infty} \frac{f^{(r)}(\alpha)}{r!} (z-\alpha)^r$$

$$= \sum_{r=0}^{n-1} \frac{f^{(r)}(\alpha)}{r!} (z-\alpha)^r + g_n(z)\,(z-\alpha)^n,$$

where $g_n(\alpha) = \frac{f^{(n)}}{n!}(\alpha)$ and g_n is analytic on D.

7. Either a function h analytic on $\mathbb{C}$ is zero everywhere or else its zeros are isolated (Corollary 1 to Theorem 15 of Section 6.7).

In other words, if $h \neq 0$ and if $h(\alpha) = 0$, there is a punctured

8. If f and g are analytic on $\mathbb{C}$ and $f(x) = g(x)$ for all $x \in \mathbb{R}$, then $f(z) = g(z)$, for all $z \in \mathbb{C}$ (a special case of The Uniqueness Theorem, Theorem 16 of Section 6.7, given in Note 10 below)

Proof :

Let $h = f - g$.

Then $h(x) = 0$, for all $x \in \mathbb{R}$.

Therefore the zeros of h are not isolated, so, using the result in Note 7,

$$h(z) = 0, \text{ for all } z \in \mathbb{C},$$

so

$$f(z) = g(z), \text{ for all } z \in \mathbb{C}.$$

9. The generalization of a Taylor series from real values to comdex values gives a unique analytic extension of the real function

This follows from the result of Note 8.

Therefore the following power series are valid:

$$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots ;$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots ;$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots .$$

10. If f and g are analytic on a region R, S is a subset of R with a cluster point in R, and f = g on S, then f = g on R (The Uniqueness Theorem, Theorem 16 of Section 6.7)

We are not assuming here that f and g are analytic everywhere, so we have to use the finite form of the Taylor series with the remainder, given in Note 6.

For a full discussion of this result, see Section 6.7.

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Self-Assessment Questions

1. State three properties of analytic functions.

2. Which of the following functions are analytic?

(i) $z \to \bar{z}$; (ii) $z \to |z|$; (iii) $z \to \text{Re } z$; (iv) $z \to \text{Im } z$.

3. Given that D is the domain of the function f where f is defined by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n,$$

and D contains more than one point, but $D \neq \mathbb{C}$, which of the following statements are true?

(i) D must be an open disc.

(ii) D must be a closed disc.

(iii) D must be a disc.

(iv) $D \supseteq \{z : |z| < k\}$, for some $k \in \mathbb{R}$.

(v) $D \supseteq \{z : |z-\alpha| < k\}$, for some $k \in \mathbb{R}$.

4. Given that f is defined by a Taylor series, $f(z) = \sum_{r=0}^{\infty} a_r z^r$, give the main steps of the proof that f is analytic on D, the inside of its circle of convergence.

5. Prove that

$$f: z \longrightarrow \begin{cases} z \sin 1/z, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

cannot be analytic on any region containing the origin.

6. Assuming the result that

$$\cosh^2 x - \sinh^2 x = 1, \quad x \in \mathbb{R},$$

why can we deduce that

$$\cosh^2 z - \sinh^2 z = 1, \quad z \in \mathbb{C}?$$

Solutions

1. There are obviously many possible answers. These include those given in the theorems of Note 1.

2. None of them. See Theorem 6 of Unit 3.

3. (iii) and (v). We cannot be sure about points on the circle of convergence. They have to be examined in each specific case.

4. Step 1: f_n is continuous (Theorem 7 of Section 6.3).

   Step 2: Therefore f_n is integrable.

   Step 3: If $f_n(z) = \sum_{r=0}^{n} a_r z^r$, Theorem 6 of Section 6.3 gives, for any contour Γ in D,

   $$\int_\Gamma f(z)dz = \int_\Gamma \lim_{n\to\infty} f_n(z)dz = \lim_{n\to\infty} \int_\Gamma f_n(z)dz.$$

   Step 4: If Γ is a triangle, by Cauchy's Theorem,

   $$\int_\Gamma f_n(z)dz = 0.$$

   Step 5: Using Morera's Theorem and Step 3, we deduce that f is analytic on D.

5. $f(z) = 0$, for $z = 1/(n\pi)$, $n = \pm 1, \pm 2$, etc. and for $z = 0$.
   0 is a cluster point of the set of zeros and therefore, by the result in Note 10, since $f \neq 0$, f cannot be analytic on any region containing 0.

6. Consider the function

   $$f(z) = \cosh^2 z - \sinh^2 z - 1.$$

   We have $f(x) = 0$, for all $x \in \mathbb{R}$.
   Therefore, by the result of Note 8,

   $f(z) = 0$, for all $z \in \mathbb{C}$.

Radio Broadcast Notes

M332 Radio Programme 4

Singularities

INTRODUCTION (to be read BEFORE LISTENING TO THE PROGRAMME)

In this programme we discuss some problems which involve finding the classifying singularities of functions. The ability to find and classify singularities is often very valuable in evaluating integrals, so practice in it is important. It is not necessary for you to have read any of Unit 8 before listening to the programme.

You should follow the Notes below during the programme. You might like to use a blank sheet of paper to cover the page and move it down to uncover each box as it is discussed. This may help prevent your eye wandering ahead of the programme. There are a few blanks in the boxes, and it is intended that you should fill these during the programme, so please have a pencil ready. As usual, there are some self-assessment questions to be attempted immediately after listening to the programme.

NOTES (to be followed DURING THE PROGRAMME)

Box 1

We say that a function f has a singularity at a point α if there is a punctured disc $0 < |z-\alpha| < r$ on which f is analytic, but f is not analytic on the whole disc $|z-\alpha| < r$.

Box 2

$f(z) = \frac{1}{z}$.

The domain of f is $\mathbb{C} - \{0\}$;

f is analytic on $0 < |z| < 1$ (for example);

f cannot be analytic on $|z| < 1$;

f has a singularity at 0.

Box 3

$f(z) = \dfrac{\sin z}{z}$

f has a singularity at 0.

Box 4

$$f(z) = \begin{cases} \dfrac{\sin z}{z}, & \text{if } z \neq 0 \\ 1, & \text{if } z = 0 \end{cases}$$

is analytic everywhere. (See Section 8.1.)

$z \to \dfrac{\sin z}{z}$ has a removable singularity at 0.

Box 5

We say that f has a pole at α if there is a function g, analytic on some neighbourhood of α, with $g(\alpha) \neq 0$, and an integer $m \geq 1$ such that

$$f(z) = \frac{g(z)}{(z-\alpha)^m} \quad \text{for } z \neq \alpha.$$

We call m the order of the pole.

Box 6

The function $f(z) = \dfrac{1}{z}$ has a pole of order [ ] at 0.

Box 7

Example 1 Find all the singularities of the functions f,g and h, where

$$f(z) = \frac{1}{z^2+1}, \quad g(z) = \frac{1}{z^2(z-1)}, \quad h(z) = \frac{1}{z^3-z^2+4z-4}.$$

Box 8

The singularities of f are at $\pm i$.

Box 9

The singularities of g are at [ ]

Box 10

Finding the singularities of h is set as an SAQ to be done after the programme.

Hint: to find the roots of $z^3 - z^2 + 4z - 4$, try $z =$ [ ]

Box 11

Example 2 Classify the singularities of f,g and h, where

$$f(z) = \frac{1}{z^2+1}; \quad g(z) = \frac{1}{z^2(z-1)}; \quad h(z) = \frac{1}{z^3-z^2+4z-4};$$

give the orders of any poles.

Box 12

$$f(z) = \frac{1}{z^2+1} = \frac{1}{z-i} \cdot \frac{1}{z+i}$$

$$= \frac{\phi(z)}{z-i}, \text{ where } \phi(z) = \frac{1}{z+i}.$$

ϕ is analytic on a neighbourhood of i and $\phi(i) \neq 0$.

Thus f has a simple pole at i.

Now $f(z) = \dfrac{\psi(z)}{z+1}$, where $\psi(x) = \dfrac{1}{z-i}$.

ψ is analytic on a neighbourhood of -i and $\psi(-i) \neq 0$.

Thus f has a simple pole at -i.

Box 13

The function $g(z) = \dfrac{1}{z^2(z-1)}$ has a [ ] at 1.

Box 14

The function g has a [ ] at 0.

Box 15

Example 3 Find, and classify, the singularities of the function cosec (where $\text{cosec} = \dfrac{1}{\sin}$).

Box 16

The singularities of cosec are at $n\pi$, $n \in \mathbb{Z}$ ($\mathbb{Z}$ being the set of integers).

Box 17

$\dfrac{\sin z}{z}$ has a removable singularity at 0, which is removed by giving it the value 1 there.

So, if

$$f(z) = \begin{cases} \dfrac{\sin z}{z}, & \text{if } z \neq 0 \\ 1, & \text{if } z = 0 \end{cases},$$

f is analytic on a neighbourhood of 0 and $f(0) \neq 0$.

Box 18

$\text{cosec } z = \frac{1}{\sin z} = \frac{z}{\sin z} \cdot \frac{1}{z} = \frac{1}{f(z)} \cdot \frac{1}{z} = \frac{g(z)}{z}$,

where $g = \frac{1}{f}$. Now g is analytic on a neighbourhood of 0 and $g(0) \neq 0$, so cosec has a simple pole at 0.

The classification of the other singularities of cosec is set as a self-assessment question to be done after the programme.

Box 19

Theorem 3

(Section 8.1)

If f has a singularity at α, and $\lim_{z\to\alpha} (z-\alpha)^m f(z)$ exists and is non-zero, then α is a pole of f of order m.

Box 20

Let $f(z) = \text{cosec } z$;

$\lim_{z\to 0} z \cdot f(z) = \lim_{z\to 0} z \text{ cosec } z = \lim_{z\to 0} \frac{z}{\sin z} = 1.$

So, by Theorem 3, cosec has a simple pole at 0.

Box 21

Example 4 Classify the singularities at 0 of the functions $f(z) = z \text{ cosec } z$ and $g(z) = \frac{1}{z} \text{ cosec } z$.

Box 22

Theorem 1

(Section 8.1)

If f has a singularity at α and $\lim_{z\to\alpha} (z-\alpha) f(z) = 0$, then α is a removable singularity of f.

Box 23

$\lim_{z\to 0} z\, g(z) = \lim_{z\to 0} \text{cosec } z$ does not exist

$\lim_{z\to 0} z^2 g(z) = \lim_{z\to 0} z \text{ cosec } z = 1.$

Box 24

So g has a pole of order [ ] at 0.

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Box 25

Example 5 Find the order of the pole at 0 of the function

$$f(z) = \frac{1}{z(1-\cos z)}.$$

Box 26

$\lim_{z\to 0} z\, f(z)$ does not exist. If $\lim_{z\to 0} z^2 f(z)$ exists, then by l'Hopital's rule

$\lim_{z\to 0} z^2 f(z) = \lim_{z\to 0} \frac{z}{1-\cos z} = \lim_{z\to 0} \frac{1}{\sin z}$.

But the last limit does not exist and so $\lim_{z\to 0} z^2 f(z)$ does not exist.

$\lim_{z\to 0} z^3 f(z) = \lim_{z\to 0} \frac{z^2}{1-\cos z} = \lim_{z\to 0} \frac{2z}{\sin z} = \lim_{z\to 0} \frac{2}{\cos z} = 2$

(using l'Hopital's rule twice). So f has a pole of order 3 at 0.

Self-Assessment Questions

1. If you have not done so already, fill in the blanks in Boxes 6, 9, 10, 13, 14, 24.

2. Define singularity and pole of order m.

3. Find and classify the singularities of the function $h(z) = 1/(z^3 - z^2 + 4z - 4)$. Use the hint in Box 10.

4. Classify the singularities of cosec at $n\pi$ where n is any non-zero integer.

5. State the theorems giving criteria for α to be a removable singularity of the function f, or a pole of f of order m (Theorems 1 and 3 of Section 8.1).

6. Classify the singularities of the function

$$f(z) = \frac{(z-\pi/2)}{z^2\cos z}$$

Solutions

1. Box 6 : one
   Box 9 : 0,1
   Box 10: 1
   Box 13: simple pole
   Box 14: pole of order 2
   Box 24: 2

2. See Boxes 1 and 5.

3. Using the hint of Box 10,

$$z^3-z^2+4z-4 = (z-1)(z^2+4) = (z-1)(z-2i)(z+2i)$$

So $h(z) = \dfrac{1}{(z-1)(z-2i)(z+2i)}$,

and has singularities at $1, 2i, -2i$.

Using reasoning similar to that in Box 12, each of these three singularities of h is a simple pole.

4. $$\lim_{z\to n\pi} (z-n\pi)\operatorname{cosec} z = \lim_{z\to n\pi} \frac{(z-n\pi)}{\sin z}$$

$$= \lim_{z\to n\pi} \frac{1}{\cos z}, \text{ by l'Hopital's rule}$$

$$= (-1)^n \neq 0.$$

Therefore, by Theorem 3 (in Box 19) cosec has a simple pole at $n\pi$.

5. See Boxes 19 and 22.

6. The singularities occur at 0 and $\frac{\pi}{2}$.

$$\lim_{z\to 0} z^2 \cdot f(z) = \lim_{z\to 0} (z - \tfrac{\pi}{2})/\cos z$$

$$= -\frac{\pi}{2} \neq 0.$$

Therefore, by Theorem 3 (in Box 19), 0 is a pole of f of order 2.

$$\lim_{z\to \pi/2} (z-\tfrac{\pi}{2})\cdot f(z) = \lim_{z\to \pi/2} \frac{(z-\frac{\pi}{2})^2}{z^2\cos z}$$

$$= \lim_{z\to \pi/2} \frac{2(z-\frac{\pi}{2})}{2z\cos z - z^2\sin z}, \text{ by l'Hopital's rule,}$$

$$= 0.$$

Therefore, by Theorem 1 (in Box 22), $\frac{\pi}{2}$ is a removable singularity of f.

Television Broadcast Notes

M332 Television Programme 5

Evaluation of Contour Integrals

INTRODUCTION (to be read BEFORE WATCHING THE PROGRAMME)

In this programme we examine several methods for evaluating contour integrals. We then look at the justification for the most general of these methods - Laurent's Theorem - and compare it with Taylor's Theorem. We aim to help you to see clearly the main stages in the proofs of these two very important theorems, and to emphasize their similarities and differences. The programme is an introductory programme, so you do not need to have studied Unit 8 before watching it.

Preparatory Question

The main results from earlier units mentioned in the programme are Cauchy's Theorem, Cauchy's Formulas, Taylor's Theorem and the Deformation Lemma. If you have time before the programme, try to write these out, or at least remind yourself of them by reading on.

Solution

See Note 1 (ii) and (iii) for Cauchy's Theorem and Cauchy's Formulas.

Taylor's Theorem (Theorem 11 of Unit 6): if f is analytic on an open disc $D = \{z : |z - \alpha| < k\}$, then the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z-\alpha)^n$ converges to f(z) for all z in D.

The Deformation Lemma (of Section 5.5) : Let C be the circle $\{\zeta : |\zeta - \alpha| = r\}$ and C' be the circle $\{\zeta : |\zeta - z| = \rho\}$ where $|z - \alpha| + \rho < r$ (so C' lies inside C). Let g be analytic on some region containing C and its inside except possibly for z. Then

$$\int_C g(\zeta)d\zeta = \int_{C'} g(\zeta)d\zeta.$$

As usual, the last section of these notes contains some self-assessment questions for you to answer after watching the programme.

NOTES

1. Methods for Evaluating the Contour Integral $\int_C f(\zeta)d\zeta$ (where C is the unit circle)

   (i) Parametrization

   Put $\zeta = e^{i\theta}$, $0 \le \theta \le 2\pi$, and 'substitute'.

   The integral then reduces to a complex-valued real integral.

   In practice, we seldom have to use this method: the other methods usually provide a short cut.

   (ii) Use of Cauchy's Theorem

   Cauchy's Theorem for star regions (Theorem 3 of Section 5.3) states: Let f be a function analytic on a star region R, and let Γ be any closed contour in R. The $\int_\Gamma f = 0$.

Example:

$$\int_C \exp(\zeta)\, d\zeta = 0.$$

N.B. This method applies only to analytic functions.

(iii) Use of Cauchy's Formula

Cauchy's Formula For Derivatives (Theorem 8 of Section 5.5) states: Let f be analytic on a region R, C a circle such that C and its inside are contained in R, n any positive integer. Then for any z inside C, f is n-times differentiable at z, and

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

Thus, Cauchy's formula leads to a method of evaluating integrals of the form $\int_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}\, d\zeta$, where the nth derivative of f is known.

Example 1:

To evaluate

$$\int_C \frac{\cos \zeta}{\zeta}\, d\zeta,$$

we put z=0, n=0, and f=cos in the formula, giving

$$\int_C \frac{\cos \zeta}{\zeta}\, d\zeta = \frac{2\pi i}{0!} \cos 0 = 2\pi i.$$

Example 2:

To evaluate

$$\int_C \frac{\cos \zeta}{\zeta^2}\, d\zeta,$$

we put z=0, f=cos, n=1 in the formula, which gives

$$\int_C \frac{\cos \zeta}{\zeta^2}\, d\zeta = \frac{2\pi i}{1!} \cdot (-\sin 0) = 0.$$

N.B. This method is appropriate for integrands which have a singularity inside the contour : that is, for some α inside the contour and some real number r, the integrand is analytic on the punctured open disc $0 < |z-\alpha| < r$, but not on the open disc $|z-\alpha| < r$.

(iv) Use of Taylor Series

From Cauchy's formulas (with z=0)

$$\int_C \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta = 2\pi i \frac{f^{(n)}(0)}{n!},$$

we see that

$$\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta = \frac{f^{(n)}(0)}{n!} \qquad (*)$$

and the term on the right is the coefficient of z^n in the Taylor series for f about the origin. Therefore, if we know the Taylor series about the origin, we can use it to evaluate the integral.

Example:

To evaluate

$$\int_C \frac{e^{\zeta}\sin\zeta}{\zeta^3}\,d\zeta,$$

we use the familiar series

$$e^z = 1 + z + \frac{z^2}{2!} + \dots ,$$

$$\sin z = z - \frac{z^3}{3!} + \dots ,$$

and so we get

$$e^z \sin z = (1 + z + \dots)\,(z - \frac{z^3}{3!} + \dots)$$

$$= z + z^2 + z^3\,(-\frac{1}{3!} + \frac{1}{2!}) + \dots \quad .$$

The coefficient of z^2 is 1.

Substituting in (*), we get

$$\frac{1}{2\pi i}\int_C \frac{e^{\zeta}\sin\zeta}{\zeta^3}\,d\zeta = 1,$$

so

$$\int_C \frac{e^{\zeta}\sin\zeta}{\zeta^3}\,d\zeta = 2\pi i.$$

(v) Use of Laurent Series

We use the following theorem (a special case of Theorem 7 of Section 8.7) : if f is a function which is analytic on the punctured disc $0<|z|<r$ and f has a singularity at the origin, then f has a Laurent series,

$$f = \dots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \dots ,$$

where

$$\int_C f(\zeta)d\zeta = 2\pi i\, a_{-1},$$

with C a circle, centre the origin, radius less than r.

Example:

To evaluate

$$\int_C \frac{1}{e^{\zeta}-1}\,d\zeta, \text{ where C is the unit circle,}$$

we cannot use Cauchy's formulas directly. However, the integrand satisfies the conditions of the above theorem, so we use the Laurent series

$$\frac{1}{e^z-1} = (z + \frac{z^2}{2!} + \dots)^{-1}$$

$$= \frac{1}{z}\,(1 + \frac{z}{2!} + \dots)^{-1}$$

$$= \frac{1}{z}\,(1 - (\frac{z}{2!} + \dots) + (\frac{z}{2!} + \dots)^2 - \dots)$$

$$= \frac{1}{z} - \dots \quad .$$

This gives

$$\int_C \frac{1}{e^{\zeta}-1}\,d\zeta = 2\pi i.$$

N.B. This method can also be used for the earlier examples and is in fact a useful generalization of the above methods.

Self-Assessment Questions

1. Evaluate the following contour integrals, where C is the unit circle.

   (i) $\int_C \sin\zeta\,d\zeta$;

   (ii) $\int_C \frac{\sin\zeta}{\zeta^2}\,d\zeta$;

   (iii) $\int_C \frac{\sin\zeta\cosh\zeta}{\zeta^3}\,d\zeta$;

   (iv) $\int_C \frac{1}{(\cos\zeta)-1}\,d\zeta$.

2. Answer the following five questions first for the proof of Taylor's Theorem on expressing the function f as a Taylor Series centred at 0 and then for the proof of the theorem of Note 1(v) on the expression of the function f as a Laurent Series about 0.

   (i) On what regions do we require f to be analytic?

   (ii) How do we use Cauchy's formula to express f(z) in terms of one or more integrals?

   (iii) What do we expand $\frac{1}{\zeta-z}$ in powers of?

   (iv) What fact do we use to prove that the remainder of this expansion tends to zero?

   (v) What expression for one or more coefficients of this series is useful in evaluating contour integrals?

Solutions

1. (i) By Cauchy's Theorem,

$$\int_C \sin \zeta \, d\zeta = 0.$$

(ii) One can use either the Laurent Series or Cauchy's Formula.

Using Cauchy's formula:

$$\int_C \frac{\sin \zeta}{\zeta^2} d\zeta = \frac{2\pi i}{1!} \cos 0$$

$$= 2\pi i.$$

(iii) One can use either the Laurent Series or the Taylor Series and Cauchy's formula.

Using the Taylor Series,

$$\sin z \cosh z = (z - \frac{z^3}{3!} + \ldots)(1 + \frac{z^2}{2!} + \ldots)$$

$$= z + z^3(\ldots).$$

So from Cauchy's formula

$$\frac{1}{2\pi i} \int_C \frac{\sin \zeta \cosh \zeta}{\zeta^3} d\zeta = \text{coefficient of } z^2 \text{ in the above}$$

$$= 0,$$

so $$\int_C \frac{\sin \zeta \cosh \zeta}{\zeta^3} d\zeta = 0.$$

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(iv) Using the Laurent Series, we get

$$\frac{1}{(\cos z)-1} = (-\frac{z^2}{2!} + \frac{z^4}{4!} - \ldots)^{-1}$$

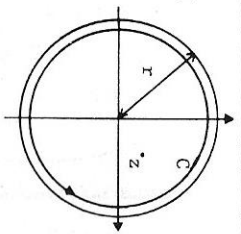
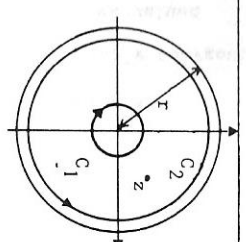
$$= -\frac{2}{z^2}(1 - \frac{2z^2}{4!} + \ldots)^{-1}.$$

The coefficient of $\frac{1}{z}$ is 0, so

$$\int_C \frac{1}{(\cos \zeta)-1} d\zeta = 0.$$

2. See the table in Note 2.

2. Proof of the Expression of a Function as a Laurent Series about Zero Compared with that of the Expression of a Function as a Taylor Series about zero.

	Taylor	Laurent
f analytic on:	$\lvert z\rvert<r$	$0<\lvert z\rvert<r$ and not necessarily $\lvert z\rvert<r$
Use Cauchy's formula.	$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta$ 	$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}d\zeta$ 
Expand $\frac{1}{\zeta - z}$ in powers of	$\frac{z}{\zeta}$ $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ + remainder	$\frac{z}{\zeta}$ for C_2 ; $\frac{\zeta}{z}$ for C_1 $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ + remainder (1) $+ a_{-1}\frac{1}{z} + a_{-2}\frac{1}{z} + \ldots + a_{-n}\frac{1}{z^n}$ + remainder (2)
Remainder tends to zero	because $\lvert\frac{z}{\zeta}\rvert < 1$	because $\lvert\frac{z}{\zeta}\rvert < 1$ for remainder (2) and $\lvert\frac{\zeta}{z}\rvert<1$ for remainder (1).
Tidy up	$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta^{n+1}}d\zeta$	$2\pi i a_{-1} = \int_C f(\zeta)d\zeta$

Later in the course these ideas are developed to deal with the integration of functions with several singularities round more complicated contours.

M332 Television Programme 6

Cauchy's Theorem and the Maximum Principle

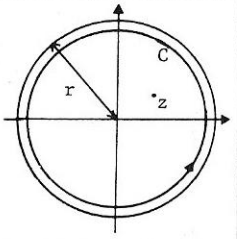
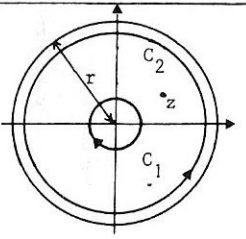
INTRODUCTION (to be read BEFORE WATCHING THE PROGRAMME)

In the first part of the programme we discuss the chain of results leading from Cauchy's Theorem for a triangle (the Cauchy-Goursat Theorem of Unit 5) to Cauchy's Theorem for simply-connected regions (Cauchy's Theorem of Unit 9). Cauchy's Theorem is one of the most important results of this course and many of the earlier results have been leading up to it. This is therefore perhaps a good moment for some revision and, if you have time before the programme, you might like to spend a few minutes writing out the definitions and results listed in the preparatory questions below. If you are short of time, just read through the solutions to remind yourself of these items, which are referred to in the programme.

The second part of the programme discusses the Maximum Principle (Theorem 1 of Unit 9 and its Corollary). There is a mistake in the result of the Maximum Principle when it first appears on the screen. The result should read

$$\begin{cases} f \text{ analytic on } R, \\ f \text{ continuous on } \bar{R}, \end{cases} \Rightarrow |f| \text{ has a maximum on boundary.}$$

The modulus signs in the last line were omitted in the broadcast.

	Taylor	Laurent
f analytic on:	$\|z\|<r$	$0<\|z\|<r$ and <u>not</u> necessarily $\|z\|<r$
Use Cauchy's formula.	$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta$ 	$f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}d\zeta$ 
Expand $\frac{1}{\zeta - z}$ in powers of	$\frac{z}{\zeta}$ $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ + remainder	$\frac{z}{\zeta}$ for C_2 ; $\frac{\zeta}{z}$ for C_1 $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ + remainder (1) $+ a_{-1}\frac{1}{z} + a_{-2}\frac{1}{z} + \ldots + a_{-n}\frac{1}{z^n}$ + remainder (2)
Remainder tends to zero	because $\|\frac{z}{\zeta}\| < 1$	because $\|\frac{z}{\zeta}\| < 1$ for remainder (2) and $\|\frac{\zeta}{z}\|<1$ for remainder (1)
Tidy up	$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta^{n+1}}d\zeta$	$2\pi i a_{-1} = \int_C f(\zeta)d\zeta$

Later in the course these ideas are developed to deal with the integration of functions with several singularities round more complicated contours.

Preparatory Questions

1. Define the following terms:

   (i) star region; (ii) polygonal line; (iii) connected set;

   (iv) polygonally-connected set; (v) polygon; (vi) simple polygon;

   (vii) maximum of $|f|$ on a region R, where f is a complex-valued function on R.

2. State the following results:

   (i) Cauchy's Theorem for a triangle (the Cauchy-Goursat Theorem);

   (ii) Cauchy's Theorem for star regions;

   (iii) the connection between connected and polygonally-connected sets;

   (iv) Gauss's Mean Value Theorem;

   (v) the Estimation Theorem for integrals;
   if f is analytic on a region R and $|f|$ is constant on an open disc D in R, what can you deduce about f on D and on R?

Solutions

1. (i) R is a star region if R is a region and if there is a point $z_0 \in R$ such that for all $z \in R$ the line segment $[z_0,z]$ is contained in R.

   (ii) A polygonal line from z to z' is a set of the form $[z,z_1] \cup [z_1,z_2] \cup \ldots \cup [z_{n-1},z']$ where $[z,z_1]$, $[z_1,z_2]$,...,$[z_{n-1},z']$ are line segments.

   (iii) An open set G is connected if it is not disconnected, i.e. if there are no open sets G_1 and G_2 such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \phi$.

   (iv) A set G is polygonally-connected if for any two points z, z' in G there is a polygonal line joining them which is contained in G.

   (v) A polygon is a closed polygonal line (or contour). (Page 69 of Unit 9)

   (vi) A simple polygon is a simple-closed polygon. (Page 69 of Unit 9).
   The point $z=\alpha$ is a maximum of $|f|$ on R if

$$|f(\alpha)| \geq |f(z)| \quad \text{for all } z \in R.$$

2. (i) Let f be analytic on a region R and let Δ be a triangular contour in R whose inside is also contained in R. Then $\int_\Delta f = 0$. (Theorem 1 of Unit 5.)

   (ii) Let f be any function analytic on a star region R, and Γ be any closed contour in R. Then $\int_\Gamma f = 0$. (Theorem 3 of Unit 5.)

   (iii) Any open set G is connected if and only if it is polygonally-connected. (Corollary to Theorem 7, and Theorem 8 of Unit 2.)

   (iv) Let f be analytic on a region containing the closed disc $|z - \alpha| \leq r$. Then $f(\alpha) = \frac{1}{2\pi} \int_0^{2\pi} f(\alpha+re^{i\theta})d\theta$. (Problem 4 of Section 5.6.)

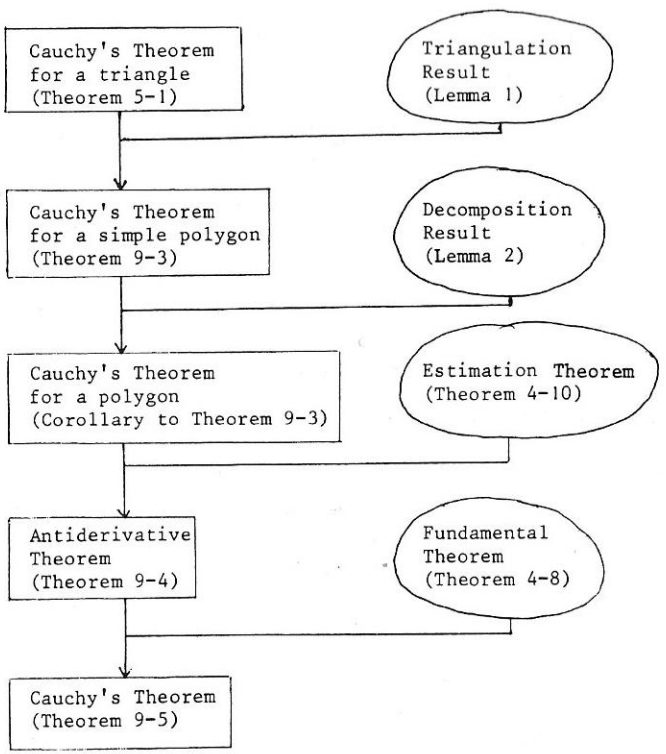
   (v) Let Γ be a contour and f a function continuous on Γ. Then $|\int_\Gamma f| \leq ML$, where L is the length of Γ and M is an upper bound of $|f|$ on Γ. (Theorem 10 of Unit 4.)

   (vi) By Theorem 17 of Unit 3, f is constant on D; by the Uniqueness Theorem (Theorem 16 of Unit 6) f is constant on R.

NOTES

1. Cauchy's Theorem

In the first part of the programme we showed how we can extend Cauchy's Theorem for star regions to Cauchy's Theorem for simply-connected regions. The following diagram summarizes the main steps. The Lemmas are from Section 9.3.


Cauchy's Theorem for a triangle (Theorem 5-1)
Triangulation Result (Lemma 1)
Cauchy's Theorem for a simple polygon (Theorem 9-3)
Decomposition Result (Lemma 2)
Cauchy's Theorem for a polygon (Corollary to Theorem 9-3)
Estimation Theorem (Theorem 4-10)
Antiderivative Theorem (Theorem 9-4)
Fundamental Theorem (Theorem 4-8)
Cauchy's Theorem (Theorem 9-5)

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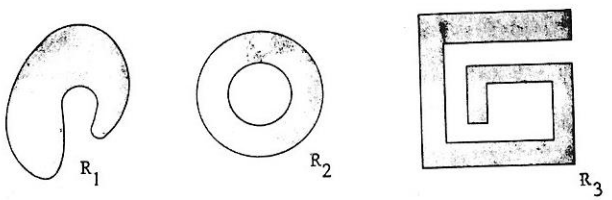
2. The Maximum Principle

In the second part of the programme we proved the Local Maximum Principle by showing that if $|f|$ has a local maximum at some point inside R then f is constant on R; in other words, if f is not constant on R then $|f|$ does not have a local maximum at a point inside R. The details can be found in Section 1 of Unit 9. Finally we asked you to try and draw diagram which illustrates the results leading from the result to the Maximum Principle. The diagram is given in the solution to Self-Assessment Question 9.

Self-Assessment Questions

1. To which of the regions shown below does Cauchy's Theorem apply?



2. State the Antiderivative Theorem.

3. Let f be analytic on a simply-connected region R; if a,b are two points in R and if Γ_1 and Γ_2 are two polygonal paths from a to b, what result justifies the statement that

$$\int_{\Gamma_1} f = \int_{\Gamma_2} f,$$

in other words, that the value of the integral is independent of the path chosen?

4. Draw a **diagram** illustrating the results leading from Cauchy's Theorem for a triangle (the Cauchy-Goursat Theorem of Unit 5) to Cauchy's Theorem (Cauchy's Theorem for Simply-Connected Regions).

5. What do we mean by saying that the (complex) function f has a maximum at α on the region R?

6. State the Maximum Principle.

7. State the Local Maximum Principle.

8. Suppose that the function f is analytic on a region R and has a local maximum at α in R. Then there is a disc centred at α on which f has a maximum at α. Let z be in this disc and let $z = \alpha + re^{i\theta}$.

   Justify each of the following steps in the proof of the Local Maximum Principle.

   Step 1: $f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha+re^{i\theta})d\theta.$

   Step 2: $\frac{1}{2\pi}\left|\int_0^{2\pi} f(\alpha+re^{i\theta})d\theta\right| \leq \frac{1}{2\pi}\int_0^{2\pi}|f(\alpha+re^{i\theta})|d\theta.$

   Step 3: $\frac{1}{2\pi}\int_0^{2\pi}|f(\alpha+re^{i\theta})|d\theta \leq \frac{1}{2\pi}\int_0^{2\pi}|f(\alpha)|d\theta.$

   Step 4: $\int_0^{2\pi}(|f(\alpha)| - |f(\alpha+re^{i\theta})|)d\theta = 0$, so $|f(\alpha)| = |f(\alpha+re^{i\theta})|$, whenever $0 \leq \theta \leq 2\pi$.

9. Draw a **diagram** illustrating the results leading from the Local Maximum Principle to the Maximum Principle.

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Solutions

1. R_1 and R_3. R_2 is not simply-connected.

2. Let f be analytic on a simply-connected region R. Then f has an antiderivative on R; that is, there is a function f analytic on R such that $F'(z) = f(z)$, $z \in R$. (Theorem 4 of Section 9.3.)

3. Cauchy's Theorem for a polygon, because $\Gamma_1 \cup (-\Gamma_2)$ is a polygon, so $\int_{\Gamma_1} f + \int_{-\Gamma_2} f = 0$.

4. See Note 1.

5. $|f(\alpha)| \geq |f(z)|$ for all $z \in R$.

6. See Note 2.

7. See Note 2.

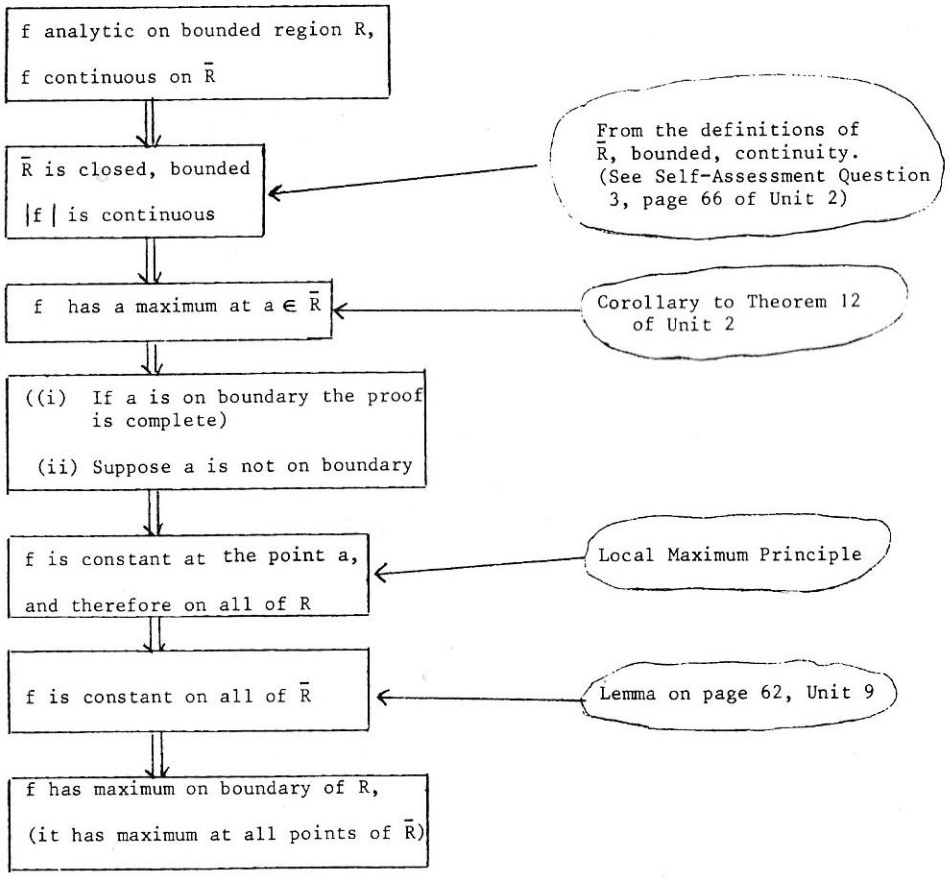
8. Step 1: from the Gauss Mean Value Theorem (see Preparatory Question 2(iv) above).

   Step 2: from the Estimation Theorem (see Preparatory Question 2(v) above).

   Step 3: because α is a local maximum.

   Step 4: because the integrand is a continuous, non-negative, real function of θ.

9. The diagram for the proof of the Maximum Principle.


f analytic on bounded region R,
f continuous on $\bar{R}$
$\bar{R}$ is closed, bounded
|f| is continuous
From the definitions of $\bar{R}$, bounded, continuity. (See Self-Assessment Question 3, page 66 of Unit 2)
f has a maximum at a ∈ $\bar{R}$
Corollary to Theorem 12 of Unit 2
((i) If a is on boundary the proof is complete)
(ii) Suppose a is not on boundary
f is constant at the point a, and therefore on all of R
Local Maximum Principle
f is constant on all of $\bar{R}$
Lemma on page 62, Unit 9
f has maximum on boundary of R,
(it has maximum at all points of $\bar{R}$)

M332 Radio Programme 5
Calculating Residues

Broadcast Notes

INTRODUCTION (to be read BEFORE LISTENING TO THE PROGRAMME)

This programme concentrates on the first two sections of Unit 10. In particular, we show you by means of various examples how to evaluate residues and how to use the Residue Theorem to evaluate certain simple contour integrals. The main results which are used are covered in the Preparatory Questions below: we suggest that if you have time you write them out; otherwise, just read through the solutions which follow.

Preparatory Questions

1. What is the Taylor series about 0 for e^{2z}?
2. What is the Taylor series about 0 for sin ?
3. State the Residue Theorem.

Solutions

1. $e^{2z} = 1+2z+(2z)^2/2! + (2z)^3/3! + (2z)^4/4! + \ldots , z \in \mathbb{C}.$
2. $\sin z = z - z^3/3! + z^5/5! - \ldots , z \in \mathbb{C}.$
3. The Residue Theorem (Theorem 11 of Section 9.7):

   Let R be a simply-connected region, $\alpha_1, \ldots, \alpha_n$ be points of R, and f be a function analytic on R except for singularities at $\alpha_1, \ldots, \alpha_n$ (so that f is analytic on $R - \{\alpha_1, \ldots, \alpha_n\}$). Let Γ be any simple-closed contour in R not passing through any of $\alpha_1, \ldots, \alpha_n$. Then

$$\int_\Gamma f(z)dz = 2\pi i \times \Big(\sum_{j \in S} \mathrm{Res}(f, \alpha_j) \Big),$$

   where $S = \{j : 1 \le j \le n \text{ and } \alpha_j \text{ lies inside } \Gamma\}$. In other words,

$$\int_\Gamma f(z)dz = 2\pi i \times (\text{the sum of the residues of f at those singularities of f lying inside } \Gamma).$$

NOTES (to be followed DURING THE PROGRAMME)

1. Definition of residue

If f is a function with singularity at α, and is analytic on a punctured disc centred at α, then on this punctured disc we can write f as a Laurent series in $z - \alpha$;

$$f(z) = \ldots + a_{-2}/(z-\alpha)^2 + a_{-1}/(z-\alpha) + a_0 + a_1(z-\alpha) + \ldots ,$$

where $a_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z-\alpha)^{n+1}}\, dz$, so that $a_{-1} = \frac{1}{2\pi i} \int_\Gamma f$.

a_{-1} is the residue of f at α.

2. Some Methods of Calculating Residues

(i) Writing out the Laurent Series Explicitly

Example 1: Find the residue of $z \to \frac{e^{2z}}{z^6}$ at 0.

$$e^{2z} = 1 + 2z + (2z)^2/2! + \ldots + (2z)^5/5! + (2z)^6/6! + \ldots$$

(see Preparatory Question 1 above),

so

$$\frac{e^{2z}}{z^6} = \frac{1}{z^6} + \ldots + \frac{2^5}{5!z} + \frac{2^6}{6!} + \ldots .$$

The coefficient of $1/z$ (the required residue) $= 2^5/5! = 4/15$.

(ii) Change of Variable

Example 2: Find the residue of $z \to e^{2z}/(z-2)^6$ at 2.

Put $z-2 = h$ (so the residue is just the coefficient of $1/h$ in the Laurent series about 0);

$$\frac{e^{2z}}{(z-2)^6} = \frac{e^{2h+4}}{h^6} = e^4 \frac{e^{2h}}{h^6} .$$

Using the result of Example 1, therefore, the residue is $e^4 . \frac{4}{15}$.

Example 3: Find the residue at 0 of $z \to \frac{1}{\sin^3 z}$.

$\sin z = z - z^3/3! + \ldots$ (see Preparatory Question 2 above),

so

$$\frac{1}{\sin^3 z} = (z - z^3/6 + \ldots)^{-3}$$
$$= z^{-3}(1 - (z^2/6 + \ldots))^{-3}$$
$$= z^{-3}(1 + z^2/2 + \ldots) .$$

The residue is the coefficient of $1/z$, which is $\frac{1}{2}$.

(iv) The Cover-Up Rule (for simple poles)

The Cover-Up Rule: Cover up the term $(z-\alpha)$, and substitute $z=\alpha$ in what's left.

Example 4: Find the residues of $z \to \frac{z+3}{(z-2)(z^2+1)}$ at its poles.

(a) The simple pole at 2:

Covering-up $(z-2)$, and putting $z=2$ in what remains gives $(2+3)(2^2+1) = 5/5 = 1$, which is the required residue.

(b) The simple poles at i and -i:

We have

$$z \to \frac{z+3}{(z-2)(z^2+1)} = \frac{z+3}{(z-2)(z-i)(z+i)} .$$

Covering-up $(z-i)$ and putting $z=i$ in what remains gives $\frac{i+3}{(i-2)\,2i}$, which simplifies to $\frac{i-1}{2}$, the required residue.

Similarly, the residue at $-i$ is $\frac{-i-1}{2}$.

(v) The g/h rule (for simple poles)

The g/h rule: If g/h has a simple pole at α, its residue there is $g(\alpha)/h'(\alpha)$.

Example 5: Find the residue of $z \to \frac{z^2}{\cos z}$ at its simple pole at $\frac{\pi}{2}$.

Taking $g(z) = z^2$, $h(z) = \cos z$, $\alpha = \frac{\pi}{2}$,

we get

$$\frac{g(\alpha)}{h'(\alpha)} = \frac{\alpha^2}{-\sin \alpha} = -\frac{\pi^2}{4} .$$

and so

$$\frac{g(\pi/2)}{h'(\pi/2)} = \frac{(\pi/2)^2}{-\sin(\pi/2)}$$

$$= -\frac{\pi^2}{4}, \text{ which is the required residue.}$$

For more examples on the g/h rule, see some of the Self-Assessment Questions in Section 10.1 of Unit 10.

3. Use of Residues to Calculate Integrals

The Residue Theorem (Theorem 11 of Unit 9 stated precisely in Preparatory Question 3 above) says that you can integrate a function f along a simple-closed contour by finding the residue of f at all its singularities inside the contour, adding these residues and multiplying the result by $2\pi i$:

(i) Circular Contours

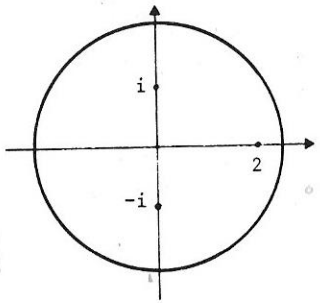
Example 6: Evaluate $\int_C \frac{e^{2z}}{(z-2)^6}\,dz$, where C is the circle $|z| = 3$.

The integrand has only one singularity inside C, which is a pole of order 6 at 2, and its residue (from Example 2) is $e^4\frac{4}{15}$.

Therefore,

$$\int_C \frac{e^{2z}}{(z-2)^6}\,dz = 2\pi i\, e^4 \frac{4}{15}$$

Example 7: Evaluate $\int_C \frac{z+3}{(z-2)(z^2+1)}\,dz$, where C is the circle $|z| = 3$.



There are three singularities of the integrand inside C: they are simple poles at 2, i and -i.

The residues (from Example 4) are

$$1, \frac{i-1}{2}, \frac{-i-1}{2}, \text{ respectively.}$$

So,

$2\pi i \times$ (the sum of the residues) $= 2\pi i \times 0 = 0$.

Therefore,

$$\int_C \frac{z+3}{(z-2)(z^2+1)}\,dz = 0.$$

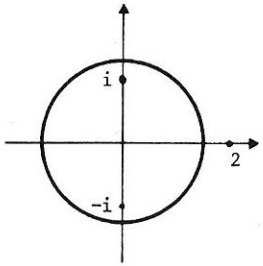
N.B.: The same function integrated around a different contour may lead to a different answer.

For example, suppose we evaluate

$$\int_{C'} \frac{z+2}{(z-2(z^2+1)}\,dz, \text{ where C' is the circle } |z| = 3/2.$$

The poles inside C' are at i and -i,

so,



$$\int_{C'} \frac{z+2}{(z-2)(z^2+1)}\,dz = 2\pi i \times$$

(residue at i + residue at -i)

$$= -2\pi i.$$

(ii) Trigonometric Real Integrals

(of the form $\int_0^{2\pi}$ (something involving cos θ, sin θ) dθ).

We replace the trigonometric integral by a contour integral round the unit circle using the "parametrization in reverse",

$z = e^{i\theta}$, so $1/z = e^{-i\theta}$.

Then,

$$\cos\theta = \tfrac{1}{2}(e^{i\theta} + e^{-i\theta}) = \tfrac{1}{2}(z + 1/z),$$

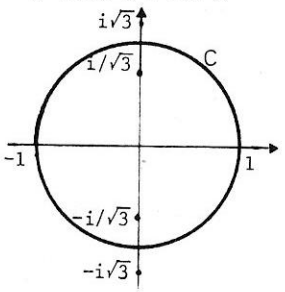
$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - 1/z),$$

$$\frac{dz}{d\theta} = i\,e^{i\theta} = iz.$$

Example 8: Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{4\cos^2\theta + \sin^2\theta}\,d\theta$.

Using the "substitution" $e^{i\theta} = z$, as explained above, gives

$$\int_0^{2\pi} \frac{1}{4\cos^2\theta + \sin^2\theta}\,d\theta = \int_C \frac{1}{(z+1/z)^2 - \frac{1}{4}(z-1/z)^2}\,\frac{dz}{iz}$$, where C is the unit circle,



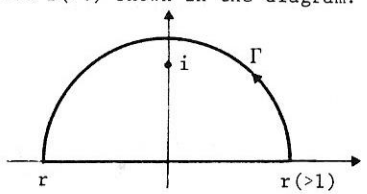
$$= \frac{4}{i}\int_C \frac{z\,dz}{(z^2+3)(3z^2+1)}$$

$$= \frac{4}{i}\cdot 2\pi i \cdot (\text{residue at } i/\sqrt{3} + \text{residue at } -i/\sqrt{3})$$

$$= \pi.$$

(iii) Semi-Circular Contour

Example 9: Evaluate $\displaystyle\int_\Gamma \frac{e^{iz}}{z^2+1}\,dz$, where Γ is the semi-circular contour of radius $r(>1)$ shown in the diagram.



The poles of the integrand are at i, -i; only the first of these lies inside the contour.

Using the g/h rule, we get

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the residue at $i = \dfrac{\exp(i^2)}{2i} = \dfrac{e^{-1}}{2i}$.

So $\displaystyle\int_\Gamma \frac{e^{iz}}{z^2+1}\,dz = 2\pi i \cdot \frac{e^{-1}}{2i} = \frac{\pi}{e}$.

(Note that this answer is independent of r.)

Self-Assessment Questions

1. Find the residue of $z \to \frac{e^{2z}}{z^6}$ at 0.

2. Find the residue of $z \to \frac{e^{2z}}{(z-2)^6}$ at 2.

3. Find the residue of $z \to \frac{1}{\sin^3 z}$ at 0.

4. Find the residue of $z \to \frac{z+3}{(z-2)(z^2+1)}$ at all of its poles.

5. Find the residue of $z \to \frac{z^2}{\cos z}$ at $\frac{\pi}{2}$.

6. Evaluate $\int_C \frac{e^{2z}}{(z-2)^6}\,dz$, where C is the circle $|z| = 3$.

7. Evaluate $\int_C \frac{z+3}{(z-2)(z^2+1)}\,dz$, where (i) C is the circle $|z| = 3$;
(ii) C is the circle $|z| = 3/2$.

8. Evaluate $\int_0^{2\pi} \frac{1}{4\cos^2\theta + \sin^2\theta}\,d\theta$.

9. Evaluate $\int_\Gamma \frac{e^{iz}}{z^2+1}\,dz$.

Solutions

1-9. See the corresponding examples in the Notes above. For Example 8, we need the residues of $z \to \frac{z}{(z^2+3)(3z^2+1)}$ at $\frac{i}{\sqrt{3}}$ and $\frac{-i}{\sqrt{3}}$ which, from the cover-up rule, are both $-1/16$.

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M332 Radio Programme 6

Uniform Convergence

INTRODUCTION (to be read BEFORE LISTENING TO THE PROGRAMME)

In this programme we start by spending some time discussing the definition of uniform convergence in a slightly different way from that in Unit 11. In the remainder of the broadcast we look at the first few theorems of Unit 11. You should follow the notes below during the programme and, attempt the Self-Assessment Questions immediately afterwards.

NOTES (To be followed DURING THE PROGRAMME)

Box 1

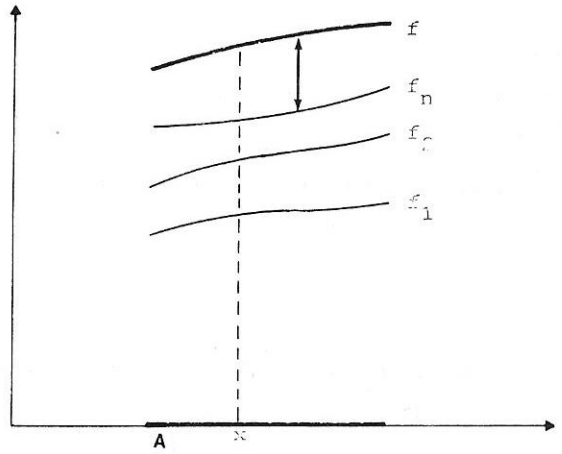
Definition

Let $\{f_n\}$ be a sequence of functions defined on a set A, and let f be a function which is also defined on A. Then f is called the uniform limit of $\{f_n\}$ on A if for every $\varepsilon > 0$ there is some N (independent of z) such that, for all z in A,

if $n > N$ then $|f(z) - f_n(z)| < \varepsilon$.

Box 2

Real Functions



Pointwise convergence means $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in A$.

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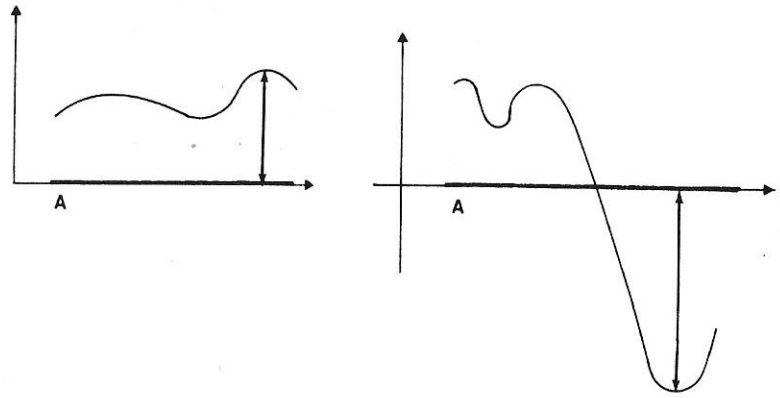
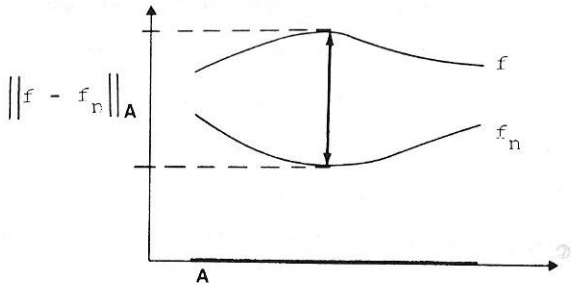
Box 3

'Size' of a function



Let $||\phi||_A = \max |\phi(x)|$, $x \in A$. We call $||\phi||_A$ the norm of ϕ on the set A.

Box 4



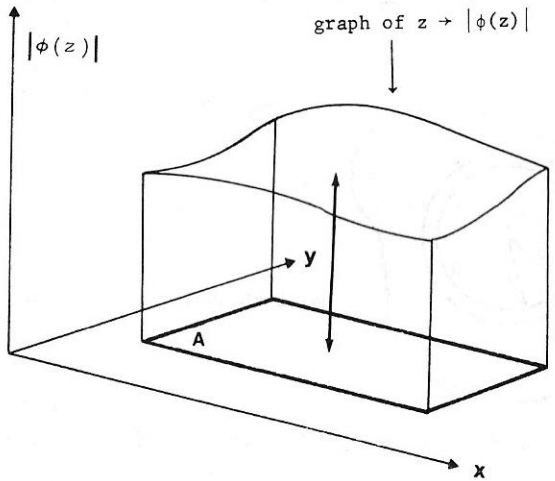
$\|f - f_n\|_A = \max|f(x) - f_n(x)|$, $x \in A$.

Uniform convergence of f_n to f on A is equivalent to

$\lim_{n\to\infty} \|f - f_n\|_A = 0$.

Box 5

Complex Functions



Size of a function

$$\|\phi\|_A = \sup|\phi(z)|,\ z \in A.$$

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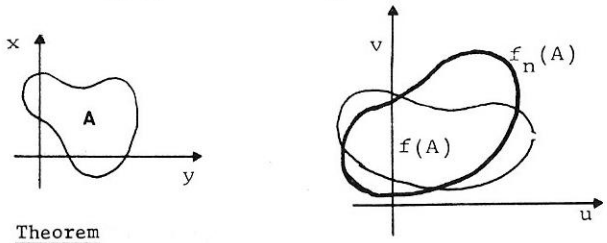
Box 6

Distance between functions

$$\|f - f_n\|_A = \sup|f(z) - f_n(z)|,\ z \in A.$$



Theorem

The sequence of functions $\{f_n\}$ converges uniformly to f on A if and only if $\lim_{n\to\infty} \|f - f_n\|_A = 0$.

Box 7

Theorem 1 of Unit 11

Suppose that $\{f_n\}$ is a sequence of functions which can be integrated along a contour Γ, and that $\{f_n\}$ converges uniformly on Γ to a function f which can be integrated along Γ. Then

$$\int_\Gamma f = \lim_{n\to\infty} \int_\Gamma f_n .$$

Proof

$$\left|\int_\Gamma f - \int_\Gamma f_n\right| = \left|\int_\Gamma (f - f_n)\right|$$

$$\leq \|f - f_n\| \cdot L \quad \text{(by the Estimation Theorem),}$$

where L is the length of the contour.

Box 8

Theorem 2 of Unit 11

Suppose that $\{f_n\}$ is a sequence of functions which are each continuous on a set A, and that $\{f_n\}$ converges uniformly to f on A. Then f is also continuous on A.

Box 9

Theorem 3 of Unit 11

Let R be a region, and let $\{f_n\}$ be a sequence of functions each analytic on R. If $\{f_n\}$ converges uniformly to f on every closed disc contained in R, then

(i) f is analytic on R;

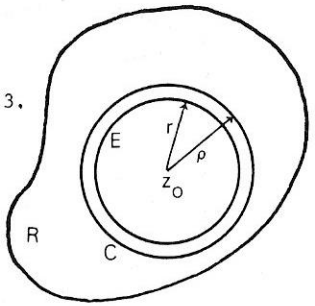
furthermore,

(ii) $\{f_n'\}$ converges uniformly to f' on every closed disc in R.

Proof

(i) See Self-Assessment Question 3.

(ii)



$$f_n'(z) = \frac{1}{2\pi i} \int_C \frac{f_n(w)}{(w-z)^2}\, dw$$

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2}\, dw.$$

So $$|f'(z) - f_n'(z)| = \left| \frac{1}{2\pi i} \int_C \frac{f(w)-f_n(w)}{(w-z)^2}\, dw \right|.$$

Self-Assessment Questions

1. Prove the following results (where, for a real function f, $||f||$ is as defined in Box 3):

   (a) $\|f\|_A \geq 0$;

   (b) $\|f\|_A = 0 \Longrightarrow f = 0$ on A;

   (c) $\|-f\|_A = \|f\|_A$;

   (d) $\|f + g\|_A \leq \|f\|_A + \|g\|_A$.

2. Prove

   (a) the equivalence which is stated in Box 4

   (b) Theorem 2 of Box 8.

3. Fill in the missing details in the proof of Theorem 3 of Box 9.

Solutions

1. (a) $\|f\|_A = \max |f(x)|,\ x \in A.$

$\geq |f(x)|,\ x \in A,$

$\geq 0.$

(b) $\|f\|_A = 0$ implies that $\max |f(x)| = 0$ for all $x \in A$, and hence that $|f(x)| = 0$ for all $x \in A$. Therefore $f = 0$ on A.

(c) $\|f\|_A = \max |f(x)|,\ x \in A,$

$= \max |-f(x)|,\ x \in A,$

$= \|-f\|_A.$

(d) For all $x \in A$

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|$$

$$\leq \max |f(x)| + \max |g(x)|,\ x \in A,$$

$$\leq \|f\|_A + \|g\|_A.$$

Since this is true for all $x \in A$,

$$\max |f(x) + g(x)| \leq \|f\|_A + \|g\|_A,$$

that is

$$\|f + g\|_A \leq \|f\|_A + \|g\|_A.$$

2. (a) Suppose $\lim_{n\to\infty} \|f - f_n\| = 0$. Let $\epsilon > 0$.

There is some N such that, for all $n > N$, we have

$$\|f - f_n\| < \epsilon.$$

Thus, if $n > N$, we have

$$\max |f(x) - f_n(x)| < \epsilon,\ x \in A,$$

and so, in particular

$$|f(x) - f_n(x)| < \epsilon,\ x \in A.$$

Hence f is the uniform limit of $\{f_n\}$ on A. Conversely, suppose f is the uniform limit of $\{f_n\}$ on A; then whenever $\epsilon > 0$ there is some number N such that, if $n > N$, we have

$$|f(x) - f_n(x)| < \epsilon, \quad \text{for all } x \text{ in } A.$$

Since this inequality is true for all $x \in A$, it follows that if $n > N$, then

$$\max |f(x) - f_n(x)| < \epsilon,$$

that is

$$\|f(x) - f_n(x)\|_A < \epsilon \text{ for all } x \in A,$$

i.e.

$$\lim_{n\to\infty} \|f - f_n\|_A = 0.$$

(b) See the solution of Problem 4 from Section 11.2.

3. See Theorem 3 of Section 11.1.

M332 Television Programe 7

Contour Integration

INTRODUCTION (To be read BEFORE WATCHING THE PROGRAMME)

The programme introduces techniques of evaluating integrals around contours using the Residue Theorem. We evaluate improper integrals of the form $\int_{-\infty}^{\infty} f(t)dt$, as well as a rather more difficult example which arises in statistics. In order to refresh your memory of the definitions of improper integrals and some of the other results we use, we suggest you try the following Preparatory Questions before watching the programme. If you are short of time, you should simply read through the solutions.

Preparatory Questions

1. State the Residue Theorem and the Estimation Theorem.
2. Let f be a complex-valued function of a real variable.
   a) What do we mean by the improper integral $\int_{-\infty}^{\infty} f(t)dt$?
   b) If f is continuous on the interval $[a,b]$ except at the point c ($a < c < b$), what do we mean by the improper integral $\int_a^b f(t)dt$?
3. Calculate the residue of the function $z \longrightarrow \frac{e^{iz}}{z^2+1}$ at $z = i$.

Solutions

1. The Residue Theorem

   Let R be a simply-connected region, $\alpha_1, \alpha_2, \ldots, \alpha_n$ be points of R, and f be a function analytic on R except for singularities at $\alpha_1, \ldots, \alpha_n$. Let Γ be any simple-closed contour in R not passing through any of $\alpha_1, \ldots, \alpha_n$. Then

$$\int_\Gamma f(z)dz = 2\pi i \sum_{j \in S} \text{Res}\,(f,\alpha_j),$$

where S = {j; $1 \le j \le n$ and α_j lies inside Γ}. In other words

$$\int_\Gamma f(z)dz = 2\pi i \times (\text{sum of residues inside } \Gamma).$$

The Estimation Theorem

Let Γ be a contour and f a function continuous on Γ. Then

$$\left| \int_\Gamma f \right| \le ML,$$

where L is the length of Γ and M is an upper bound for $|f|$ on Γ.

2. a) if f is a complex-valued function of a real variable, we define the improper integral $\int_{-\infty}^{\infty} f(t)dt$ to be the number defined by the formula

$$\int_{-\infty}^{\infty} f(t)dt = \lim_{r\to\infty} \int_{-r}^{r} f(t)dt.$$

provided that this limit exists.

b) If f is a complex-valued function of a real variable, and f is continuous at all points of the interval [a,b] except at the point c, then the improper integral $\int_a^b f(t)dt$ is the number defined by the formula

$$\int_a^b f(t)dt = \lim_{\epsilon\to 0^+} \left(\int_a^{c-\epsilon} f(t)dt + \int_{c+\epsilon}^b f(t)dt \right),$$

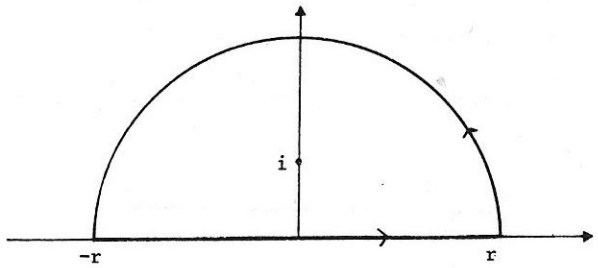
provided that this limit exists.

3. See the Broadcast Notes to Radio Programme 5, Example 9.

NOTES

Some of the algebra in the programme is rather intricate, but do not worry if you cannot follow it at the time: a full account is given in the notes.

1. The first example was introduced in Radio Programme 5. In this example we wish to evaluate $\int_\Delta \frac{e^{iz}}{z^2+1}\,dz$, where the contour Δ consists of the real axis from -r to r (r>1), together with the semicircular part $\cap$ in the upper half-plane as shown below.



Since r > 1, the only pole of $z \longrightarrow \frac{e^{iz}}{z^2+1}$ which lies inside this contour is a simple pole at z = i.

By Preparatory Question 3, the residue at this point is 1/(2ei). Hence, by the Residue Theorem,

$$\int_\Delta \frac{e^{iz}}{z^2+1}\,dz = 2\pi i \text{ Res } (f,i)$$
$$= \pi/e .$$

We can obtain more information from this integral by splitting it into two integrals - one along the real axis and the other along the semicircle. What we shall prove is that the integral along the semicircle has limit zero as r becomes large. We do this by using the Estimation Theorem. We need to find an upper bound for $\left|\frac{e^{iz}}{z^2+1}\right|$ on Δ. Now

$$|e^{iz}| = |e^{i(x+iy)}|$$
$$= |e^{ix}||e^{-y}|$$
$$= |e^{-y}|, \text{ since } |e^{ix}| = 1.$$

But y is positive on the semicircle and so

$$|e^{iz}| = |e^{-y}| \le 1.$$

Also, using the triangle inequality, we have

$$|z^2+1| \ge \big|\, |z|^2 - 1 \big|$$
$$= r^2 - 1$$

on $\cap$. Hence, by the Estimation Theorem,

$$\left| \int_\cap \frac{e^{iz}}{z^2+1}\,dz \right| \le ML \text{ where } M = \frac{1}{r^2-1},\ L = \pi r,$$

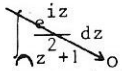
$$= \frac{1}{r^2-1}\,\pi r$$

Thus

$$\pi/e = \int_{\Delta} \frac{e^{iz}}{z^2+1}\,dz$$

$$= \int_{-r}^{r} \frac{e^{it}}{t^2+1}\,dt + \int_{\cap} \frac{e^{iz}}{z^2+1}\,dz$$

where the first integral is written in terms of t, rather than z, since we are integrating along the real axis. In the programme we wrote



to indicate that this integral has limit zero as r becomes large, but you must always remember to prove this as we have done above.

Hence, using the definition of improper integral in Preparatory Question 2a,

$$\pi/e = \lim_{r\to\infty} \int_{-r}^{r} \frac{e^{it}}{t^2+1}\,dt$$

$$= \int_{-\infty}^{\infty} \frac{e^{it}}{t^2+1}\,dt .$$

Equating real and imaginary parts yields two further equalities,

$$\int_{-\infty}^{\infty} \frac{\cos t}{t^2+1}\,dt = \pi/e \quad , \quad \int_{-\infty}^{\infty} \frac{\sin t}{t^2+1}\,dt = 0.$$

By starting with a contour integral, we have evaluated two real integrals; in a similar way we can integrate various other functions of this kind along a semicircle to get analogous results. We usually take the integral to be a rational function. The main steps of the method are itemized below.

To evaluate $\int_{-\infty}^{\infty} f(t)\begin{Bmatrix}\cos\beta t\\ \sin\beta t\end{Bmatrix}dt$, consider $\int_{\Delta} f(z)e^{i\beta z}dz$, where Δ is a semicircular contour in the upper-half plane.

Then use

(1) the Residue Theorem to evaluate the contour integral

(2) the Estimation Theorem to show that $\int_{\cap} f(z)e^{i\beta z}dz$ tends to zero as r becomes large, and deduce the value of $\int_{-\infty}^{\infty} f(t)e^{i\beta t}dt$.

(3) Equate real and imaginary parts to obtain

$$\int_{-\infty}^{\infty} f(t)\cos\beta t\,dt \text{ and } \int_{-\infty}^{\infty} f(t)\sin\beta t\,dt.$$

Normally, however, we turn the above procedure around and start with a real integral and use a contour integral to help us evaluate it.

2. One problem that can arise is when the integrand has a singularity on the real axis as in the second example of the programme:

$\int_{-\infty}^{\infty} \frac{\cos t}{(t^2+1)(t-2)}\,dt$. The integral is clearly an improper integral since the integrand is not defined for t = 2, and the point 2 lies on the contour; to get around this we separate the integral into two parts

$$\int_{-\infty}^{\infty} \frac{\cos t}{(t^2+1)(t-2)}\,dt = \lim_{r\to\infty} \int_{-r}^{r} \frac{\cos t}{(t^2+1)(t-2)}\,dt.$$

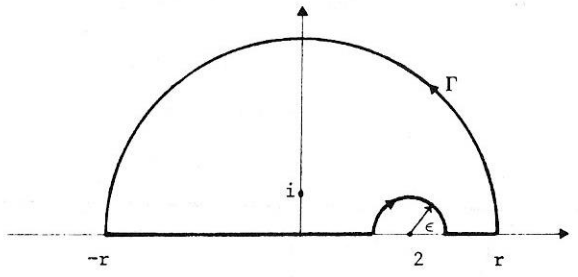
$$= \lim_{r\to\infty} \int_{-r}^{2-\epsilon} \frac{\cos t}{(t^2+1)(t-2)}\,dt + \lim_{r\to\infty} \int_{2+\epsilon}^{r} \frac{\cos t}{(t^2+1)(t-2)}\,d$$

and then take the limit as ϵ approaches zero from above.

That is, we calculate

$$\lim_{\substack{r\to\infty\\ \epsilon\to 0^+}} \left(\int_{-r}^{2-\epsilon} + \int_{2+\epsilon}^{r} \right) \frac{\cos t}{(t^2+1)(t-2)}\,dt$$

The semicircular contour Γ that we use to evaluate this integral must avoid the point 2 and so we make a semicircular indentation at the point 2,



and evaluate $\int_{\Gamma} \frac{e^{iz}}{(z^2+1)(z-2)}\,dz$. The only singularity of the integrand which lies inside Γ is the pole at z = i, and so

$$\int_{\Gamma} \frac{e^{iz}}{(z^2+1)(z-2)}\,dz = 2\pi i \operatorname{Res}(i),$$

where we have used Res(i) in place of Res(f,i) since, in this problem, no confusion can arise as to which function the residue refers. As before let $\frown$ denote the semicircular part of Γ and let $\Gamma(\epsilon)$ denote the semicircular indentation around 2 taken clockwise. We may then write the integral as follows.

$$\int_{\frown} \frac{e^{iz}}{(z^2+1)(z-2)}\,dz = \int_{-r}^{2-\epsilon} + \int_{\Gamma(\epsilon)} + \int_{2+\epsilon}^{r} + \int_{\Gamma} \frac{e^{iz}}{(z^2+1)(z-2)}\,dz$$

We now evaluate

i) $\lim_{\epsilon\to 0^+} \int_{\Gamma(\epsilon)} \frac{e^{iz}}{(z^2+1)(z-2)}\,dz.$

and ii) $\lim_{r\to\infty} \int_{\frown} \frac{e^{iz}}{(z^2+1)(z-2)}\,dz.$

i) Using the result of Problem 4 of Section 10.5 of Unit 10 we have

$$\lim_{\epsilon\to 0^+} \int_{\Gamma(\epsilon)} \frac{e^{iz}}{(z^2+1)(z-2)}\,dz = -\pi i \text{ Res}(2)$$

ii) Using the Estimation Theorem, we can show that

$$\left| \int_{\frown} \frac{e^{iz}}{(z^2+1)(z-2)}\,dz \right| \le \frac{1}{(r^2-1)(r-2)}\,\pi r,$$

and so

$$\lim_{r\to\infty} \int_{\frown} \frac{e^{iz}}{(z^2+1)(z-2)}\,dz = 0.$$

Combining these results, we get

$$2\pi i \text{ Res}(i) = \lim_{\substack{r\to\infty \\ \epsilon\to 0^+}} \left(\int_{-r}^{2-\epsilon} + \int_{\Gamma(\epsilon)} + \int_{2+\epsilon}^{r} + \int_{\Gamma} \right) \frac{e^{iz}}{(z^2+1)(z-2)}\,dz$$

$$= \int_{-\infty}^{\infty} \frac{e^{it}}{(t^2+1)(t-2)}\,dt - \pi i \text{ Res}(2)$$

We ask you to complete this problem in Self-Assessment Question 1 on page 56.

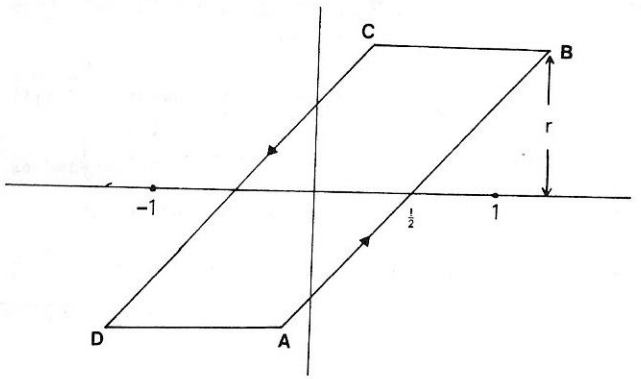
3. The final example is much harder than any you would be asked to do; it was chosen because of its importance in statistics. Do not worry if you cannot follow all the calculation, or if you cannot see why we make such a strange choice of f and of the contour. Just try and make sure that you follow the basic method using the Residue Theorem and letting the contour become large.

The example was to show that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, or equivalently that $\int_{-\infty}^{\infty} \exp(-2\pi t^2)\,dt = 1/\sqrt{2}$. This is obtained by substituting $x = t\sqrt{2\pi}$.

To do this we consider the function f defined by

$$f(z) = \frac{\exp(i\pi z^2)}{\sin \pi z}, \quad z \in \mathbb{C},$$

and integrate it along the parallelogram $\square$ shown below.



The singularities of f are simple poles at $z = 0, \pm 1, \pm 2, \ldots$, but the only one lying inside ABCD is at $z = 0$, at which the residue is $1/\pi$. Hence, by the Residue Theorem,

$$\int_{\square} f = 2\pi i \times \frac{1}{\pi}$$

$$= 2i.$$

Since $\square$ = AB + BC + CD + DA, we have

$$2i = \int_{\square} f = \int_{AB} f + \int_{BC} f + \int_{CD} f + \int_{DA} f. \qquad (1)$$

Consider $\int_{AB} f$. Parametrize AB by $z = \frac{1}{2} + (1+i)t$, $-r \le t \le r$

Clearly $\frac{dz}{dt} = (1+i)$ and so, on substituting for z and using the fact that $(1+i)^2 = 2i$, we have

$$\int_{AB} f = \int_{AB} \frac{\exp(i\pi z^2)}{\sin \pi z}\, dz$$

$$= \int_{-r}^{r} \frac{\exp(i\pi(\tfrac{1}{2} + (1+i)t)^2)}{\sin \pi(\tfrac{1}{2} + (1+i)t)} \cdot (1+i)\,dt$$

$$= \int_{-r}^{r} \frac{\exp(i\pi(\tfrac{1}{4} + (1+i)t + 2it^2)}{\cos(\pi(1+i)t)} \cdot (1+i)\,dt$$

$$= (1+i)\exp(i\pi/4) \int_{-r}^{r} \frac{\exp(i\pi(1+i)t)\ \exp(-2\pi t^2)}{\cos(\pi(1+i)t)}\, dt$$

$$= i\sqrt{2} \int_{-r}^{r} \frac{\exp(i\pi(1+i)t)\ \exp(-2\pi t^2)}{\cos(\pi(1+i)t)}\, dt.$$

Similarly we can show that

$$\int_{CD} f = i\sqrt{2} \int_{-r}^{r} \frac{\exp(-i\pi(1+i)t)\ \exp(-2\pi t^2)}{\cos(\pi(1+i)t)}\, dt,$$

and so

$$\int_{AB} f + \int_{CD} f = i\sqrt{2} \int_{-r}^{r} \frac{2\cos(\pi(1+i)t)\ \exp(-2\pi t^2)}{\cos(\pi(1+i)t)}\, dt$$

$$= 2\sqrt{2}\, i \int_{-r}^{r} \exp(-2\pi t^2)\, dt.$$

There remain the two horizontal sections BC and DA.

Parametrize BC by $z = t + (1+i)r$, $-\frac{1}{2} \le t \le \frac{1}{2}$.

On BC $|\exp(i\pi z^2)| = |\exp(i\pi(t + (1+i)r)^2)|$

$$= |\exp(i\pi(t^2 + 2(1+i)rt + 2ir^2)|$$

$$= |\exp(i\pi(t^2 + 2rt))| \cdot |\exp(-2\pi rt - 2\pi r^2)|$$

$$= |\exp(-2\pi rt - 2\pi r^2)|$$

$$\le \exp(-2\pi r^2 + \pi r).$$

Now $|\sin z| = \left|\frac{e^{iz} - e^{-iz}}{2i}\right|$

$$\ge \left|\frac{|e^{iz}| - |e^{iz}|}{2}\right|$$

$$= \left|\frac{e^{-y} - e^{y}}{2}\right|, \quad \text{where } z = x + iy,$$

So, on BC we have

$$|\sin \pi z| \ge |\sinh \pi r| > 1, \text{ for } r \text{ large.}$$

Hence $\left|\frac{\exp(i\pi z^2)}{\sin \pi z}\right| \le |\exp(-2\pi r^2 + \pi r)|$

and so, by the Estimation Theorem,

$$\left|\int_{BC} \frac{\exp(i\pi z^2)}{\sin \pi z}\, dz\right| \le |\exp(-2\pi r^2 + \pi r)|$$

which has limit zero as r becomes large.

Similarly $\int_{DA} \frac{\exp(i\pi z^2)}{\sin \pi z}\, dz$ has limit zero as r becomes large.

Hence, letting r become large in (1), we have

$$2i = 2\sqrt{2}\, i \int_{-\infty}^{\infty} \exp(-2\pi t^2)\, dt,$$

and so

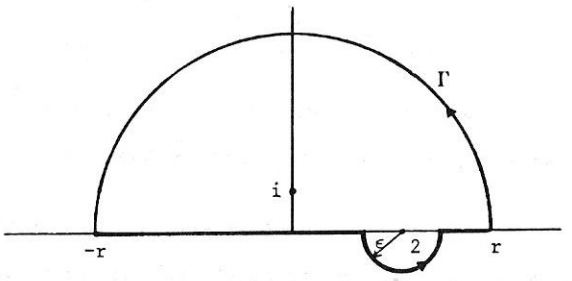
$$\int_{-\infty}^{\infty} \exp(-2\pi t^2)\, dt = 1/\sqrt{2}$$

Self-Assessment Questions

1. (i) Find the residues of the function $f: z \longrightarrow \frac{e^{iz}}{(z^2+1)(z-2)}$ at the points $z = 2$, and $z = i$, and hence evaluate $\int_{-\infty}^{\infty} \frac{e^{it}}{(t^2+1)(t-2)}\, dt$.

   (ii) Evaluate the integrals $\int_{-\infty}^{\infty} \frac{\cos t}{(t^2+1)(t-2)}\, dt$ and $\int_{-\infty}^{\infty} \frac{\sin t}{(t^2+1)(t-2)}\, dt$.

2. Show that the result of Question 1 remains unaltered if we choose the contour shown below.



Solutions

1. (i) $z \longrightarrow \dfrac{e^{iz}}{(z^2+1)(z-2)}$ has a simple pole at $z = 2$ and by the cover-up rule, the residue is $e^{2i}/5$. At $z = i$ there is a simple pole, and by the cover-up rule the residue is $e^{-1}/2i(i-2)$.

But, from Note 2,

$$2\pi i \text{ Res}(i) = \int_{-\infty}^{\infty} \frac{e^{it}}{(t^2+1)(t-2)}\, dt - \pi i \text{ Res}(2)$$

and so

$$\int_{-\infty}^{\infty} \frac{e^{it}}{(t^2+1)(t-2)}\, dt = \frac{2\pi i\, e^{-1}}{2i(i-2)} + \frac{\pi i e^{2i}}{5}$$

$$= -\frac{\pi(i+2)e^{-1}}{5} + \frac{\pi i\, e^{2i}}{5}$$

(ii) Equating real and imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{\cos t}{(t^2+1)(t-2)}\, dt = -\frac{2\pi e^{-1}}{5} - \frac{\pi}{5}\sin 2,$$

$$\int_{-\infty}^{\infty} \frac{\sin t}{(t^2+1)(t-2)}\, dt = -\frac{\pi e^{-1}}{5} + \frac{\pi}{5}\cos 2.$$

2. Let Γ denote the contour used in Question 1 and let $\Gamma(\epsilon)$ denote the circular contour, centre 2, radius ϵ, taken in an anticlockwise direction.

Then $\Gamma + \Gamma(\epsilon) = \Gamma_1$

Since the only poles of f which lie inside Γ_1 are at $z = i$ and $z = 2$, we have, by the Residue Theorem, that

$$\int_{\Gamma_1} f = 2\pi i\, (\text{Res}(2) + \text{Res}\ (i)\ ).$$

Hence

$$2\pi i(\text{Res}(2) + \text{Res}(i)) = \int_{\Gamma_1} f$$

$$= \int_{\Gamma} f + \int_{\Gamma(\epsilon)} f$$

$$= \int_{\Gamma} f + 2\pi i \text{ Res}(2).$$

Letting r become large and ϵ become small, we have

$$2\pi i(\text{Res}(2) + \text{Res}(i)) = \int_{-\infty}^{\infty} \frac{e^{it}}{(t^2+1)(t-2)} - \pi i \text{ Res}(2) + 2\pi i \text{ Res}(2)$$

and so

$$\int_{-\infty}^{\infty} \frac{e^{it}}{(t^2+1)(t-2)} = 2\pi i \text{ Res } (i) + \pi i \text{ Res}(2),$$

as before.

M332 TV 8

M332 Television Programme 8

Broadcast Notes

Analytic Continuation

INTRODUCTION (to be read BEFORE WATCHING THE PROGRAMME)

Before watching this programme it would be helpful for you to know what is meant by a function element, an analytic multi-function, and direct analytic continuation. In the programme we discuss the gamma function and show that it is analytic on the region $\{z: \mathrm{Re}\, z > 1\}$. We then obtain an analytic continuation of it to a larger region. In the next part of the programme we show, by an example, that not every analytic multi-function contains a function element with largest domain, and finally we look at an example involving a Riemann surface. To prepare for the programme, we suggest you try the following questions.

Preparatory Questions

1. Define the following terms,

   (i) function element

   (ii) direct analytic continuation

   (iii) chain of function elements

   (iv) analytic multi-function.

2. Let R be a region and Γ a contour (not necessarily in R), and let $H: \Gamma \times R \to \mathbb{C}$ be a function. What conditions do we require on the functions

   (i) $t \to H(t,z_0)$, where $z_0 \in R$ is fixed,

   (ii) $z \to H(t_0,z)$, where $t_0 \in \Gamma$ is fixed,

   (iii) H,

in order that the function ϕ defined by

$$\phi(z) = \int_\Gamma H(t,z)\,dt$$

be analytic on R?

[This is Theorem 5 of Unit 11].

Solutions

1. (i) If a function is analytic on a region R, and R is the domain of f, then the pair (f,R) is called a function element.

   (ii) The function elements (f_1,R_1) and (f_2,R_2) are said to be direct analytic continuations of each other if $f_1(z) = f_2(z)$ for all z in some non-empty open subset of $R_1 \cap R_2$.

   (iii) The finite sequence of function elements

   $$\{(f_1,R_1), (f_2,R_2), \ldots, (f_n,R_n)\}$$

   is said to form a chain if (f_{k+1},R_{k+1}) is the direct analytic continuation of (f_k,R_k) for $k = 1,2,\ldots, n-1$.

   (iv) The equivalence classes of the equivalence relation "(f,R) is related to (g,S) if there is a chain joining (f,R) to (g,S)" are called analytic multi-functions.

2. The conditions we require are:

   (i) $t \to H(t,z_0)$, where $z_0 \in R$, must be continuous on Γ;

   (ii) $z \to H(t_0,z)$, where $t_0 \in \Gamma$, must be analytic on R;

   (iii) H must be bounded on $\Gamma \times R$.

NOTES

1. Some solutions of the differential equation $F'(z) = [F(z)]^2$

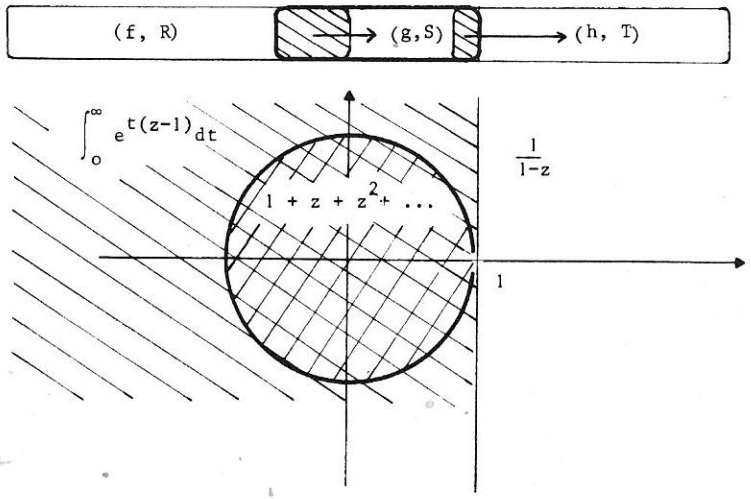
The following functions f, g and h with domains R, S and T respectively, are all solutions of the given differential equation:

$f: z \to 1 + z + z^2 + \ldots$ $\qquad R = \{z: |z| < 1\}$

$g: z \to \int_0^\infty e^{t(z-1)}\,dt$ $\qquad S = \{z: \mathrm{Re}\, z < 1\}$

$h: z \to \frac{1}{1-z}$ $\qquad T = \{z: z \neq 1\}.$

These solutions give rise to three function elements (f,R), (g,S) and (h,T). The three function elements are all solutions of the same differential equation We know that if f = g on some open subset of $R \cap S$, then f and g are direct analytic continuations of each other. Hence we can form the chain {(f,R), (g,S), (h,T)} which we can represent diagramatically as follows. Note that $R \cap S \neq \emptyset$, $S \cap T \neq \emptyset$ - this is why we get a chain.



2. The Gamma Function

The gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt, \qquad \mathrm{Re}\, z > 1.$$

We shall show that there are direct analytic continuations of the gamma function; but first, we must prove that it is analytic on Re z > 1. For Re z > 1

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt$$

$$= \left(\int_0^1 + \int_1^2 + \ldots + \int_n^{n+1} + \ldots \right) t^{z-1} e^{-t}\,dt.$$

Define the function p_n by

$$p_n(z) = \int_n^{n+1} t^{z-1} e^{-t}\,dt, \qquad \mathrm{Re}\, z > 1.$$

We show that Γ is analytic by proving that

(a) p_n is analytic for each integer n,

(b) the above series is uniformly convergent.

(a) We do this by showing that the conditions of Preparatory Question 2 are satisfied for the function defined by $H(t,z) = t^{z-1} e^{-t}$, Re z > 1 and $t \in [0,\infty)$. Thus we need to prove that

(i) for each fixed z_0, $t \to t^{z_0-1} e^{-t}$ is continuous;

(ii) for each fixed t_0, $z \to t_0^{z-1} \exp(-t_0)$ is analytic;

(iii) the function $(t,z) \to t^{z-1} e^{-t}$ is bounded.

The proof of (i) is straightforward, the function being the product of two continuous functions. The proof of (ii) follows from standard results, and so we are left to prove (iii).

Let $\alpha > 1$ and choose $z \in \mathbb{C}$ such that $1 < \mathrm{Re}z < \alpha$. Then

$$\begin{aligned} |t^{z-1} e^{-t}| &= |t^{z-1}| \cdot |e^{-t}| \\ &= |\exp((z-1)\mathrm{Log}\ t)| \cdot |e^{-t}| \\ &= |\exp((\mathrm{Re}z-1)\mathrm{Log}\ t)| \cdot |e^{-t}| \\ &\leq |t^{\alpha-1}| \cdot |e^{-t}| \\ &\leq (n+1)^{\alpha-1} e^{-n}, \end{aligned}$$

whenever $t \in [n,n+1]$. Hence the function $(t,z) \to t^{z-1} e^{-t}$ is bounded, and so p_n is analytic for each n.

(b) For each integer n,

$$\begin{aligned} |p_n(z)| &= \left| \int_n^{n+1} t^{z-1} e^{-t}\, dt \right| \\ &\leq \int_n^{n+1} |t^{z-1} e^{-t}| dt \\ &\leq (n+1)^{\alpha-1} e^{-n}. \end{aligned}$$

Now $\sum_{n=0}^{\infty} (n+1)^{\alpha-1} e^{-n}$ is convergent (by the ratio test), and therefore by the Weierstrass M-test we have that $\sum_{n=0}^{\infty} p_n(z)$ is uniformly convergent. Hence Γ is analytic on the region $1 < \mathrm{Re}z < \alpha$ and so, since α was chosen arbitrarily ($\alpha > 1$), we have that Γ is analytic on $\mathrm{Re}z > 1$.

Let Γ_1 denote the gamma function with domain $\mathrm{Re}z > 1$; we shall find a direct analytic continuation of Γ_1. For $\mathrm{Re}z > 1$ we have, upon integrating by parts, that

$$\begin{aligned} \Gamma_1(z) &= \int_0^{\infty} t^{z-1} e^{-t}\, dt \\ &= \frac{1}{z} t^z e^{-t} \Big|_0^{\infty} + \int_0^{\infty} \frac{1}{z} t^z e^{-t}\, dt \\ &= \int_0^{\infty} \frac{1}{z} t^z e^{-t}\, dt, \end{aligned}$$

since $\lim_{t \to \infty} (\frac{1}{z} t^z e^{-t}) = 0$. Therefore

$$\Gamma_1(z) = \frac{1}{z} \Gamma_1(1+z).$$

Now $z \to \frac{1}{z} \Gamma_1(1+z)$ is analytic on $\mathrm{Re}(1+z) > 1$, that is on $\mathrm{Re}z > 0$, and so we can define the function Γ_2 by

$$\Gamma_2(z) = \frac{1}{z} \Gamma_1(1+z), \quad \mathrm{Re}z > 0.$$

Clearly on $\mathrm{Re}z > 1$, $\Gamma_2(z) = \Gamma_1(z)$ and so $(\Gamma_2, \{z:\mathrm{Re}z > 0\})$ is a direct analytic continuation of $(\Gamma_1, \{z:\mathrm{Re}z > 1\})$. We can repeat this process and show that $z \to \Gamma_2(1+z)$ is analytic on $\mathrm{Re}(1+z) > 0$, that is on $\mathrm{Re}z > -1$, and then define Γ_3 by

$$\Gamma_3(z) = \frac{1}{z} \Gamma_2(1+z),$$

with domain $\{z: \mathrm{Re}z > -1$ and $z \neq 0\}$. We may proceed indefinitely in this manner to obtain a direct analytic continuation of $(\Gamma_1, \{z: \mathrm{Re}z > 1\})$ to any

$$(\Gamma_n, \{z: \mathrm{Re}z > -n \text{ (n an integer)}, n \neq 0,-1,-2,\ldots,-n+1\}).$$

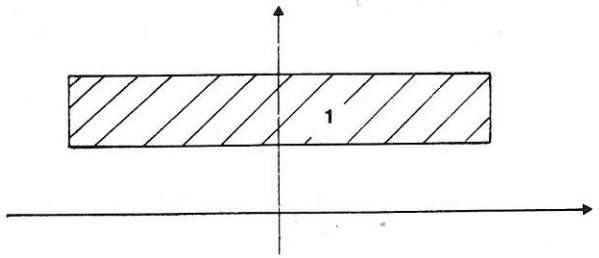
That we can find a function element with largest domain we ask you to show in Self-Assessment Question 2.

3. In the first two parts of the programme we saw examples for which we could find a function element with largest domain. We now show, by means of the following example, that this is not always possible.

Suppose the log function had a function element with largest domain; then this domain would be $\mathbb{C} - \{0\}$ and the function element would be given by

$$(\log z = \log |z| + i\theta, \ \mathbb{C} - \{0\}).$$

Also, any complete chain $\{(f_1, R_1), (f_2, R_2), \ldots, (f_{n-1}, R_{n-1}), (f_n, R_1)\}$ would have to be such that $f_1 = f_n$, since each R_i is a subset of this largest domain. Consider a function element which has the shaded area in the diagram below as domain

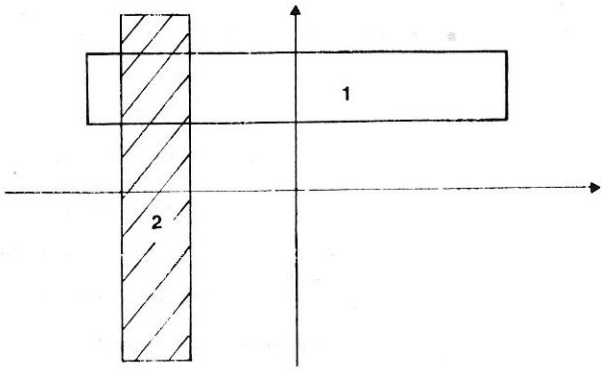


As this domain lies in the upper-half plane we know that the argument of log must satisfy the inequality

$$0 \leq \theta \leq \pi.$$

and the values in this domain will be precisely defined since it is a subset of the largest domain. We now build up a complete chain that encircles the point 0 as follows.

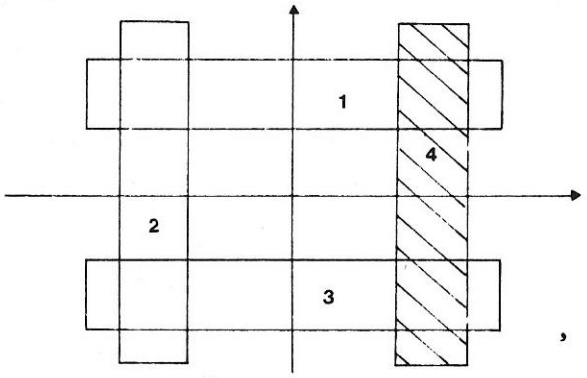
We continue this function analytically to a function element in the overlapping region shown below.



Since the regions 1 and 2 have a nonempty intersection, θ must satisfy the inequality

$$\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$$

Repeat this process twice to obtain



,

where in region 4 we must have

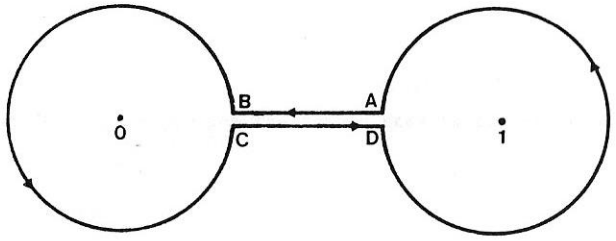
$$\frac{3\pi}{2} \leq \theta \leq \frac{5\pi}{2}$$

If we now analytically continue the function from region 4 to a region 5 which overlaps region 1 then the condition on θ is

$$2\pi \leq \theta \leq 3\pi$$

But this gives us different function values from region 1, which is a contradiction. Hence there is no function element with largest domain.

4. In the final part of the programme we showed how we could overcome the problem of the logarithm having no function element with largest domain by introducing the complete analytic logarithm function whose domain is a Riemann surface (a spiral surface). If we now repeat the analytic continuations of Note 3 we shall find that region 5 and region 1 are different sets since they are at different levels of the Riemann surface; that is, what we now have is a chain $\{(f_1,R_1), (f_2,R_2), \ldots, (f_5,R_5)\}$ which is not complete, since $R_1 \neq R_5$. Finally we showed, by means of a model of the Riemann surface $z \to \log z - \log(z-1)$, that if we have the contour



drawn on the Riemann surface, then

$$-\int_{AB} (\log z - \log(z-1))dz \neq \int_{CD} (\log z - \log (z-1))dz$$

since AB and CD are on different levels of the Riemann surface.

Self-Assessment Questions

1. The differential equation $F'(z) = [F(z) - 1]^2$

   has a series solution

   $$f: z \to z - z^2 + z^3 - z^4 + \ldots, \quad |z| < 1.$$

   This series has sum $\frac{z}{1+z}$. How do we know that $g: z \to \frac{z}{1+z}$, $z \neq 1$ is a solution of the differential equation?

2. By considering $\Gamma(z) = \phi(z) + \psi(z)$, where

   $$\phi(z) = \int_0^1 t^{z-1}e^{-t}dt$$

   $$\psi(z) = \int_1^\infty t^{z-1}e^{-t}dt,$$

   show that ψ is entire by showing that it is analytic on every closed disc $\{z: |z| \leq k\}$. Moreover, show that ϕ has a series expansion for all $z \in \mathbb{C} - \{0, -1, -2, \ldots\}$. Hence find the function element of Γ with largest domain.

3. Let $R_1 = \{z: \text{Im } z < 0\}$ and let (ϕ_1, R_1) be a function element such that $\phi_1(z) = \phi_1(z + i)$ if $z \in R_1$. Does the analytic multi-function containing (ϕ_1, R_1) also contain a function element with largest domain?

Solutions

1. The function g is analytic on $\mathbb{C} - \{1\}$ and

   $$g(z) = f(z) \text{ whenever } |z| < 1,$$

   and so g is a direct analytic continuation of f. Hence, by the Permanence of Functional Relationships, g is a solution of the differential equation.

2. We can express $\psi(z)$ as a series

   $$\psi(z) = \sum_{n=1}^{\infty} p_n(z),$$

   where $p_n(z) = \int_n^{n+1} t^{z-1}e^{-t}dt$. The proof that ψ is entire is the same

as the proof in Note 2, except that here we must show that the function $(t,z) \to t^{z-1}e^{-t}$ is bounded on $[n,n+1] \times \mathbb{C}$. Let D be the closed disc $\{z: |z| \leq k\}$ $(k > 1)$ and let $z \in D$. Whenever $t \in [n,n+1]$, $n \geq 1$, we have

$$\begin{aligned} |t^{z-1}| &= t^{\mathrm{Re}z-1} \\ &\leq t^{k-1} \\ &\leq (n+1)^{k-1}, \text{ since } t \geq 1. \end{aligned}$$

Hence $|t^{z-1}e^{-t}| \leq (n+1)^{k-1}e^{-n}$, whenever $t \in [n,n+1]$ and so the function $(t,z) \to t^{z-1}e^{-t}$ is bounded.

We may now proceed as before to show that ψ is analytic on every closed disc, and hence is entire.

Now let $\mathrm{Re}z > 1$. Then

$$\begin{aligned} |t^{z-1}| &= t^{\mathrm{Re}z-1} \\ &\leq 1 \end{aligned}$$

whenever $t \in [0,1]$. Hence the function $(t,z) \to t^{z-1}$ is bounded for $\mathrm{Re}z > 1$ and $t \in [0,1]$.

Since the series for e^{-t} is uniformly convergent and the function $(t,z) \to t^{z-1}$ is bounded in any disc $\{z: |z| \leq k\}$, the series $\sum_{n=0}^{\infty} \frac{(-t)^n}{n!} t^{z-1}$ is uniformly convergent, and so

$$\begin{aligned} \phi(z) &= \int_0^1 t^{z-1}e^{-t}dt \\ &= \int_0^1 \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} t^{z-1}dt \\ &= \sum_{n=0}^{\infty} \int_0^1 \frac{(-t)^n t^{z-1}}{n!} dt \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(z+n)\,n!} \end{aligned}$$

where the last series converges for all $z \in \mathbb{C} - \{0,-1,-2,\dots\}$.

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Hence the function element with largest domain is

$$z \to \left(\psi(z) + \sum_{n=0}^{\infty} \frac{(-1)^n}{(z+n)n!} \qquad \mathbb{C} - \{0,-1,-2\dots\} \right).$$

3. Let $R_2 = \{z: \mathrm{Im}z < 1\}$, and consider the function $\phi_2: z \to \phi_1(z+i)$, $z \in R_2$. Then, whenever $z \in R_1$, $\phi_2(z) = \phi_1(z+i) = \phi_1(z)$ and, moreover $R_1 \cap R_2$ is nonempty. Hence (ϕ_2, R_2) is a direct analytic continuation of (ϕ_1, R_1). Since this process can be continued indefinitely there is a function element with largest domain.

Radio Broadcast Notes

M332 R7

M332 Radio Programme 7

Laplace Transforms

INTRODUCTION (to be read BEFORE LISTENING TO THE PROGRAMME)

In this programme we identify the results and techniques of Unit 14 which we expect you to be able to reproduce. The unit has two main themes, Laplace transforms and asymptotic expansions, and it is the former which we discuss first.

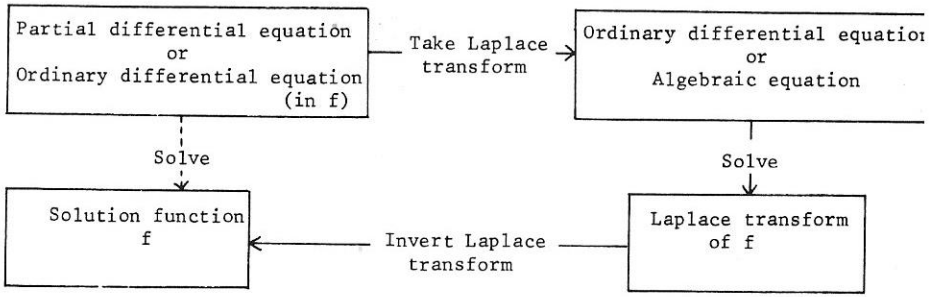
A common use for Laplace transforms is in solving differential equations. The Laplace transform turns the differential equation into an equation which can usually be solved more easily. The solution function so obtained is then the Laplace transform of the solution of the differential equation, and the problem is then to invert the above process in order to retrieve the solution. We introduce a complex integral which does this inversion and we call it 'the Inversion Integral for Laplace Transforms'. Under suitable conditions this integral can be evaluated, by the Residue Theorem.

In the second part of the programme we discuss asymptotic expansions. We use asymptotic expansions when the function in which we are interested is not specified by a formula, but by some indirect method, and the expansion approximates the behaviour of the function for large values of the variables. We are mainly interested in asymptotic expansions for integrals, and in particular in Watson's Lemma, which tells us how to work out the asymptotic expansion of a function which has the form of a Laplace transform. We use this result to develop the technique of finding asymptotic expansions by the method of steepest descent and it is this idea which we describe in the last part of the programme. The technique described in this last part is optional material. Before listening to the broadcast you will probably find it helpful to glance through the unit, noting the

definitions of the Laplace transform and asymptotic expansions.

NOTES (to be followed DURING THE PROGRAMME)

1.



2. Table of Laplace transforms

Function	Laplace transform
$f(t)$	$F(p)$
t	$1/p$, Re $p>0$
e^{at}	$1/(p-a)$, Re $p>a$
⋮	⋮

This is used in the same way as a table of standard integrals.

3. The inversion integral for Laplace transforms is given by

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt} F(p)\,dp .$$

(See Section 14.2)

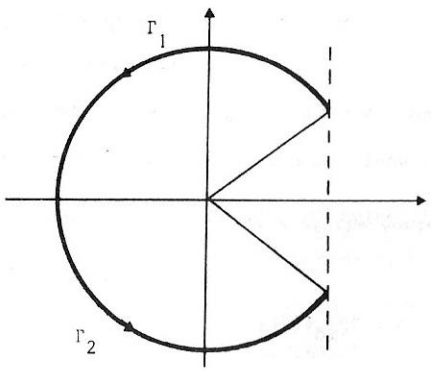
4. Theorem 6

Let Γ_1 and Γ_2 be the contours

$$\Gamma_1 = \{p = re^{i\theta} \; ; \; \frac{\pi}{2} - \alpha \leq \theta \leq \pi, \; \sin\alpha = \frac{\gamma}{r}, \quad \gamma>0\}$$

and

$$\Gamma_2 = \{p = re^{i\theta} \; : \; -\pi \leq \theta \leq \alpha - \frac{\pi}{2}, \; \sin\alpha = \frac{\gamma}{r}, \; \gamma>0\}.$$



Suppose that there are positive constants K, k, p_o such that, on the arcs Γ_1 and Γ_2,

$$|F(p)| < K|p|^{-k}, \text{ where } |p| > p_o;$$

then, for $t > 0$,

$$\lim_{r\to\infty} \int_{\Gamma_1} F(p)e^{pt}dp = \lim_{r\to\infty} \int_{\Gamma_2} F(p)e^{pt}dp = 0.$$

5. Definition

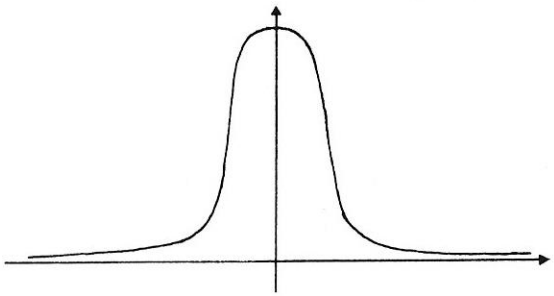
Let $S_n(t) = a_o + \frac{a_1}{t} + \ldots + \frac{a_n}{t^n}$, $n\in\mathbb{N}$, and suppose that, for every fixed n,

$$\lim_{t\to\infty} t^n(f(t) - S_n(t)) = 0;$$

then we say that $a_o + a_1/t + \ldots + \frac{a_n}{t^n} + \ldots$ is an asymptotic expansion of f, and that $S_n(t)$ is asymptotic to f(t).

M332 R7

6. The following diagram shows the graph of a real function whose integral is largely determined by a small part of its domain.



7.

Form of integral :

$$\int_\Gamma e^{ph(z)}f(z)dz, \text{ where p is real.}$$

↓

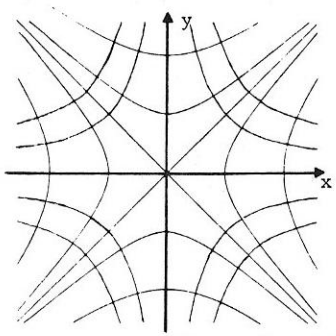
Look for the saddle points of the surface describing $(x,y) \to |h(z)|$.

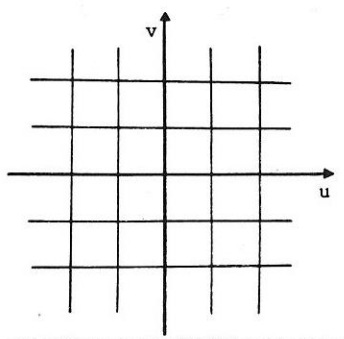
↓

These occur where $h'(z) = 0$. At a saddle point we can find u(x,y) and v(x,y), where $h(z) = u(x,y) + iv(x,y)$.

↓

h specifies a mapping from the (x,y) plane to the (u,v) plane; in the latter plane the lines u = constant, v = constant are orthogonal, and so on the (x,y) plane the lines u(x,y) = constant, v(x,y) = constant are orthogonal.





$|e^{ph(x,y)}| = |e^{p(u(x,y) + iv(x,y))}| = |e^{pu(x,y)}|.$

If $u(x,y) = a$ and $v(x,y) = b$ at the saddle point, then the path of steepest descent is given by

$$h(x,y) = a - t + ib$$

where t increases from zero.

Self-Assessment Questions

1. (i) Write down the inversion integral for the Laplace transform F.

   (ii) Draw the contour that we use to evaluate this integral.

   (iii) Write down the conditions of F in order that we can deduce (by Theorem 6)

$$\lim_{r\to\infty} \int_{\Gamma_1} F(p)e^{pt}dp = \lim_{r\to\infty} \int_{\Gamma_2} F(p)e^{pt}dp = 0,$$

   where Γ_1, Γ_2 are the curved parts of the contour of part (ii).

2. Which of the following statements are true and which are false?

   (i) The more terms we take in an asymptotic expansion the better the approximation will be.

   (ii) Asymptotic series are convergent series.

   (iii) The first term of an asymptotic expansion is called the dominant term.

   (iv) For large values of the variables, the approximation by an asymptotic expansion gets progressively better as more terms are taken until an optimum value for the number of terms is reached.

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Solutions

1. (i) $f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}F(p)dp.$

   (ii) See the figure of Note 4.

   (iii) If there are positive constants K, k and p_o such that on the curved parts of the above figure

$$|F(p)| < K|p|^{-k} \text{ where } |p| > p_o,$$

   then the integral around this curved part has limit zero.

2. (i) False. See Page 46 of Section 14.7.

   (ii) False. In general they are not convergent.

   (iii) True.

   (iv) True.

M332 TV9

M332 Television Programme 9

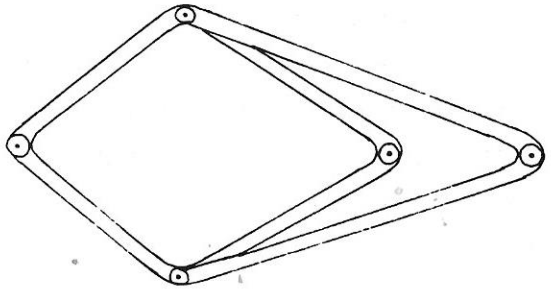
Analytic Functions

INTRODUCTION (to be read BEFORE WATCHING THE PROGRAMME)

In this programme we introduce some of the important concepts of Unit 13. We start by defining a bilinear transformation and we then investigate some of its geometric properties. In particular we look at images of circles and straight lines under bilinear transformations. In order to overcome a problem that arises in this context we define a 'new point' to be appended to the complex plane; we then demonstrate, using models, various transformations of this extended complex plane. Before watching the programme you should read the introduction to Unit 13 and you should make sure that you understand the meaning of a complex function as a transformation. You should also check that you know what is meant by: $\lim_{z \to a} f(z) = \infty$; the definition of the latter can be found in Unit 8, page 21.

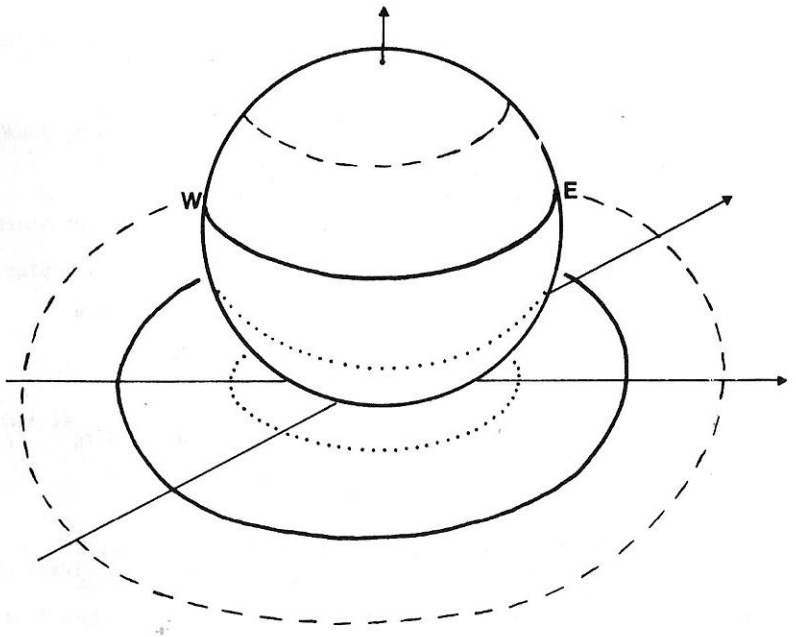
NOTES (to be read AFTER THE PROGRAMME)

The result that the image of a circle or straight line under a bilinear transformation is a circle or straight line can be found in Section 13.3. We demonstrated the special case of the transformation of inversion using Peaucellier's cell, illustrated below.



[To see why this instrument corresponds to an inverse transformation see, for example: H.M. Cundy and A.P. Rollett, Mathematical Models (Oxford University Press, 1951).]

The point at infinity and stereographic projection are covered in Section 13.5. We completed the programme by discovering which bilinear transformations of the complex plane corresponded to certain rotations of its stereographic image. The first rotation we chose was a 180° rotation about the E-W axis in the diagram below.



Now, any rotation of the sphere is conformal, as is stereographic projection, and so the corresponding transformation of the plane must be conformal. To identify which conformal transformation, we considered the images of various circles. The equator is the stereographic projection of a circle (which we denote as the unit circle) and under the rotation it maps to itself.

A circle of lattitude in the northern hemisphere is the stereographic projection of a circle outside the unit circle; under the relation this is mapped to a circle in the southern hemisphere which corresponds to a circle inside the unit circle. Hence the outside of the unit circle (corresponding to the northern hemisphere) is mapped to the inside of the unit circle (corresponding to the southern hemisphere) by this transformation. In particular the point at infinity is mapped to the origin and so the corresponding transformation of the plane is the transformation $z \to 1/z$.

Finally we looked at a rotation of the sphere about an axis which was not horizontal and we saw an animation which showed the corresponding transformation of the complex plane.

Self-Assessment Questions

1. State conditions on the constants a,b,c and d that ensure that the bilinear transformation $z \to \frac{az+b}{cz+d}$ represents
   (i) a translation
   (ii) a rotation
   (iii) a magnification
   (iv) an inversion.

2. Show that the bilinear transformation $z \to \frac{az+b}{cz+d}$ is one-one.

3. What is the transformation of the plane which corresponds to a rotation of the sphere (as used in stereographic projection) about a north-south axis?

Solutions

1 (i) if $c = 0$, $d = a = 1$, $b \neq 0$ then $z \to z+b$ is a translation.

(ii) if $c = b = 0$, $d = 1$, $|a| = 1$ then $z \to az$ is a rotation.

(iii) If $c = b = 0$, $d = 1$, $a \in R$ and $a > 1$ then $z \to az$ is a magnification. (It is a contraction if $a \in R$ and $a < 1$.)

(iv) If $a = d = 0$, $b = c = 1$ then $z \to \frac{1}{z}$ is an inversion.

Notice that the choices of (a,b,c,d) are not necessarily unique. for example, in part (iv), $a = d = 0$ and $b = c = 3$ gives exactly the same bilinear transformation.

2. Suppose that

$$\frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d},$$

then

$$0 = \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}$$

$$= \frac{(az_1+b)(cz_2+d) - (az_2+b)(cz_1+d)}{(cz_1+d)(cz_2+d)}$$

$$= \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)}.$$

Hence $z_1 = z_2$, since $ad-bc \neq 0$.

Alternatively, construct the inverse transformation.

3. Under the rotation circles of lattitude are mapped to themselves, and so corresponding circles of the plane are mapped to themselves. Hence the corresponding transformation of the plane is a rotation.

M332 TV 11

M332 Television Programme 11

Revision

INTRODUCTION (to be read BEFORE WATCHING THE PROGRAMME)

This programme is used for revision purposes. We work through three questions which are similar to the type of question you will get in the exam. In the main, the exam questions will consist of some theory followed by an application of the theory to a particular problem. The first question we consider is a straightforward application of Cauchy's Formula, and the second shows how we can prove a special case of Taylor's Theorem by starting with Cauchy's Formula and proceeding using 'common sense'. In the final example we draw a suitable contour for evaluating a particular improper integral, and we mention several instances where care should be taken. The evaluation is left as a self-assessment question to be completed after the programme. A full solution is given in the following notes.

NOTES (to be read AFTER THE PROGRAMME)

Problem 1

State Cauchy's Formula and hence evaluate

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta)\,d\theta.$$

(Hint: consider $\int_C \frac{e^z}{z}\,dz$, where C is the circle $|z| = 1$.)

Solution

Cauchy's Formula

Let f be analytic on a simply-connected region R containing a simple-closed contour Γ. Then

$$f(a) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-a}\,dz, \quad a \text{ inside } \Gamma.$$

(We would normally expect you to find a suitable complex integral for the second part of the question by using, for example, the techniques of Section 10.2 of Unit 10. However, for a complicated integrand a hint would be given.)

Consider $\int_C \frac{e^z}{z}\,dz$, where C is the unit circle.

Since exp is analytic on a simply-connected region containing C, we can apply the above formula with f = exp and a = 0 to obtain

$$\exp(0) = \frac{1}{2\pi i}\int_C \frac{e^z}{z}\,dz,$$

that is

$$2\pi i = \int_C \frac{e^z}{z}\,dz.$$

To obtain a real integral we parametrize C by $z = e^{i\theta}$, $0 \le \theta \le 2\pi$. Then, since $dz = ie^{i\theta}d\theta$, we have

$$2\pi i = \int_0^{2\pi} \frac{\exp(e^{i\theta})ie^{i\theta}}{e^{i\theta}}\,d\theta,$$

or equivalently

$$2\pi = \int_0^{2\pi} \exp(e^{i\theta})\,d\theta.$$

Now

$$\exp(e^{i\theta}) = \exp(\cos\theta + i\sin\theta)$$
$$= \exp(\cos\theta)\ \exp(i\sin\theta)$$
$$= \exp(\cos\theta)\ (\cos(\sin\theta) + i\sin(\sin\theta)),$$

and so

$$2\pi = \int_0^{2\pi} \exp(\cos\theta)(\cos(\sin\theta) + i\sin(\sin\theta))d\theta.$$

Equating real parts, we get

$$2\pi = \int_0^{2\pi} \exp(\cos\theta)\cos(\sin\theta)d\theta.$$

Remark

An equally valid statement of Cauchy's Formula is given by Theorem 7 of Unit 9, namely: Cauchy's Formula for a closed contour. In the programme the contour we used was a circle C, and the region R was a region (simply-connected) which contained C. In an exam question the version required would be stated explicitly.

Problem 2

State Taylor's Theorem, and prove the special case of Taylor's Theorem for expansions about the origin.

Solution

Taylor's Theorem

If f is analytic on an open disc $D = \{z:|z - \alpha| < k\}$, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(z - \alpha)^n$$

converges to f(z) for all z in D.

The proof of the special case $\alpha = 0$ can be found on pages 126 and 127 of

Unit 6. Points to note are that we start with Cauchy's Formula.

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w - z}\,dw,$$

and expand $\frac{1}{w-z}$ in powers of z/w using the formula

$$1 + \alpha + \alpha^2 + \ldots + \alpha^n = \frac{1 - \alpha^{n+1}}{1 - \alpha}$$

for the sum of a geometric series. We express this in the equivalent form

$$\frac{1}{1 - \alpha} = 1 + \alpha + \alpha^2 + \ldots + \alpha^n + \frac{\alpha^{n+1}}{1 - \alpha},$$

and put $\alpha = z/w$ to obtain the power series for $\frac{1}{w-z}$.

Using this series we express f(z) as a sum of n integrals of the form $\frac{z^n}{2\pi i}\int_C \frac{f(w)}{w^n}\,dw$, plus a further integral which we denote by R_n. We use Cauchy's Formula to express the first n integrals in the required form, and then show that $|R_n|$ has zero limit as n becomes large.

[Remarks

The condition for Taylor's Theorem as stated in the programme are: if f is analytic on a region R containing a point a, then the series

$$f(z) = f(a) + f'(a)(z - a) + \frac{f''}{2!}(z-a)^2 + \ldots$$

is convergent in any disc centre a in R. Since it is the disc of convergence which is important we see that the two versions are essentially the same.]

Problem 3

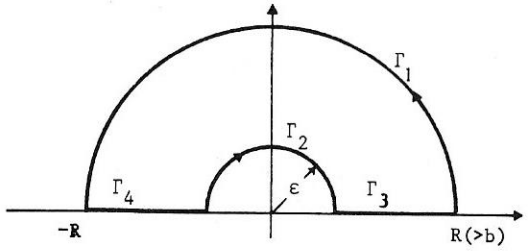
Evaluate $\int_0^{\infty} \frac{\log x}{x^2+b^2}\,dx$, where b>0.

Solution

Step 1

We associate with the given real integral a related contour integral.

Let Γ be the closed contour shown below, where Γ_1 is the semicircle of radius $R(>b)$ and Γ_2 is the semicirclar indentation at 0 of radius $\varepsilon(<b)$. Γ_3 and Γ_4 denote those parts of the contour formed by the real axis.



Consider $\int_\Gamma \frac{L_R(z)}{z^2 + b^2}\,dz$, where L_R is the branch of the logarithm associated with the fundamental region $R = \{z: -\frac{\pi}{2} < \mathrm{Im}z < \frac{3\pi}{2}\}$. Note that with this choice of the branch of the logarithm, the function $f: z \to \frac{L_R(z)}{z^2 + b^2}$ is analytic on a region containing Γ.

Step 2

We use the Residue Theorem to evaluate the contour integral around Γ.

The function f has a simple pole at $z = + ib$ lying inside Γ.

Now $$\mathrm{Res}(f,ib) = \lim_{z\to ib} \frac{L_R(z)}{z + ib}$$

$$= \frac{L_R(ib)}{2ib} = \frac{\log|ib| + i\pi/2}{2ib}.$$

By the Residue Theorem,

$$\int_\Gamma f = \frac{\pi}{b}(\log b + i\pi/2) \quad (b>0).$$

We now note that

$$\int_\Gamma f = \int_{\Gamma_1} f + \int_{\Gamma_2} f + \int_{\Gamma_3} f + \int_{\Gamma_4} f.$$

M332 TV10

Step 3

We use the Estimation Theorem to show that the integral around Γ_1 becomes arbitrarily small in modulus if we take R to be large enough.

On Γ_1; $z = Re^{i\theta}$ $0\le\theta\le\pi$, so $|L_R(z)| = |\log R + i\theta| \le \log R + |\theta| < \log R + \pi$.

Hence, by the Estimation Theorem,

$$\left|\int_{\Gamma_1} f\right| \le \frac{\log R + \pi}{R^2 - b^2}\pi R, \quad (\text{since } |z^2 + b^2| \ge \left||z^2| - |b^2|\right| = |R^2 - b^2|)$$

$$= \frac{\pi(\log R + \pi)}{R - b^2/R}.$$

Since $\lim_{x\to\infty} \frac{\log x}{x - a/x} = \lim_{x\to\infty} \frac{1/x}{1 + a/x^2} = 0$, by L'Hôpital's rule, it follows that

$$\lim_{R\to\infty} \left|\int_{\Gamma_1} f\right| = 0.$$

Step 4

We use the Estimation Theorem to show that the integral around Γ_2 becomes arbitrarily small in modulus if we take ε to be small enough. On Γ_2, $z = \varepsilon e^{i\theta}$, $0 \le \theta \le \pi$, so

$$|L_R(z)| = |\log\varepsilon + i\theta| \le \log\varepsilon + \pi.$$

Hence, by the Estimation Theorem,

$$\left|\int_{\Gamma_2} f\right| \le \frac{\log\varepsilon + \pi}{|b^2 - \varepsilon^2|}\pi\varepsilon, \quad (\text{since } |z^2+b^2| \ge |b^2-\varepsilon^2|)$$

$$= \frac{\pi(\varepsilon\log\varepsilon + \pi\varepsilon)}{b^2 - \varepsilon^2}.$$

Since $\lim_{x\to0} x\log x = -\lim_{x\to0} \frac{-\log x}{1/x} = \lim_{x\to0} \frac{1/x}{-1/x^2} = \lim_{x\to0}(-x) = 0$, again by L'Hôpital's rule, it follows that

$$\lim_{\varepsilon\to0} \left|\int_{\Gamma_2} f\right| = 0.$$

Step 5

We now express the integrals along Γ_3 and Γ_4 in terms of real integrals.

Now, on Γ_3, $z = t$, $\varepsilon \le t \le R$, and on Γ_4, $z = te^{i\pi}$, $-R \le t \le -\varepsilon$. (Note the care that must be taken with the parametrization of Γ_4.) Hence;

(i) $$\int_{\Gamma_3} f = \int_{\varepsilon}^{R} \frac{L_R(t)}{t^2 + b^2} dt = \int_{\varepsilon}^{R} \frac{\log t}{t^2 + b^2} dt.$$

(ii) $$\int_{\Gamma_4} f = \int_{-R}^{-\varepsilon} \frac{L_R(te^{i\pi})}{t^2 e^{i2\pi} + b^2} e^{i\pi} dt$$

$$= -\int_{-R}^{-\varepsilon} \frac{\log|t| + i\pi}{t^2 + b^2} dt.$$

$$= \int_{\varepsilon}^{R} \frac{\log t + i\pi}{t^2 + b^2} dt.$$

Final step

(Putting it all together).

$$\int_{\Gamma} f = \int_{\Gamma_1} f + \int_{\Gamma_2} f + \int_{\Gamma_3} f + \int_{\Gamma_4} f = \frac{\pi}{b} (\log b + i\pi/2.)$$

Let $R \to \infty$ and $\varepsilon \to 0$; then by our previous results we obtain

$$\int_0^{\infty} \frac{\log t}{t^2 + b^2} dt + \int_0^{\infty} \frac{\log t + i\pi}{t^2 + b^2} dt = \frac{\pi}{b} (\log b + i\pi/2).$$

Equating the real parts we get

$$\int_0^{\infty} \frac{\log t}{t^2 + b^2} dt = \frac{\pi \log b}{2b}.$$

This question is rather more difficult than anything you are likely to get in the exam, but it is a good exercise to follow through, and shows how, with a little care, the steps resolve themselves as you would expect.